Joint distribution of the process and its sojourn time in a half-line $[a, +\infty)$ for pseudo-processes governed by higher-order heat-type equations

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Joint distribution of the process and its sojourn time in a half-line $[a, +\infty)$ for pseudo-processes driven by a high-order heat-type equation

Valentina CAMMAROTA* and Aimé LACHAL†

Abstract

Let $(X(t))_{t\geq 0}$ be the pseudo-process driven by the high-order heat-type equation $\frac{\partial u}{\partial t} = \pm \frac{\partial^N u}{\partial x^N}$, where N is an integer greater than 2. We consider the sojourn time spent by $(X(t))_{t\geq 0}$ in $[a, +\infty)$ ($a \in \mathbb{R}$), up to a fixed time $t > 0$: $T_a(t) = \int_0^t \mathbb{1}_{[a,+\infty)}(X(s))\,\mathrm{d}s$. The purpose of this paper is to explicit the joint pseudo-distribution of the vector $(T_a(t), X(t))$ when the pseudo-process starts at a point $x \in \mathbb{R}$ at time 0. The method consists in solving a boundary value problem satisfied by the Laplace transform of the aforementioned distribution.

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1 Introduction

Let N be an integer greater than 2 and let $\kappa_{_N}=(-1)^{1+N/2}$ if N is even, $\kappa_{_N}=\pm 1$ if N is odd. Let us introduce the pseudo-process $X=(X(t))_{t\geq 0}$ related to the high-order heat-type equation

$$\frac{\partial u}{\partial t}=\kappa_{_N}\frac{\partial^N u}{\partial x^N}. \tag{1.1}$$

This pseudo-process is driven by a family of signed measures $(\mathbb{P}_x)_{x\in\mathbb{R}}$, which are not probability measures, such that for any positive integer n, any $t>0$, $0=t_0<t_1<\cdots<t_n$ and $x=x_0,x_1,\dots,x_n,y\in\mathbb{R}$,

$$\mathbb{P}_x\{X(t)\in \mathrm{d}y\}=p(t;x-y)\,\mathrm{d}y$$

and

$$\mathbb{P}_x\{X(t_1)\in \mathrm{d}x_1,\dots,X(t_n)\in \mathrm{d}x_n\}=\prod_{i=1}^{n}p(t_i-t_{i-1};x_{i-1}-x_i)\,\mathrm{d}x_i.$$

In the above definition, the function p stands for the "heat-kernel" associated with the equation (1.1). It solves (1.1) together with the initial condition $p(0;x)=\delta(x)$ and it is characterized by its Fourier transform as

$$\int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}ux}p(t;x)\,\mathrm{d}x=\begin{cases}\mathrm{e}^{-tu^N} & \text{if } N \text{ is even,}\\ \mathrm{e}^{\kappa_N t(-\mathrm{i}u)^N} & \text{if } N \text{ is odd.}\end{cases}$$

Pseudo-processes have been introduced and studied by many authors in the early 60's: some pioneering works are [6], [15], [16]. In these works, the efforts were made in providing accurate and proper definitions of pseudo-processes driven by signed, complex or vectorial measures. The heat-type equations of order 3 and 4 are of special interest since they arise, for instance, in various problems of linear elasticity.

Subsequently, more specific aspects of the pseudo-process X introduced here have been considered. In [1], [2], [4], [5], [8] to [14], [17] to [22], the authors paid attention to several classical functionals: first or last overshooting time above or below a fixed level, up-to-date maximum and minimum functionals, sojourn times in a half-line... Many explicit results are known about the pseudo-distribution of the various aforementioned functionals. In this paper, we focus on the sojourn time of X in a half-line $[a,+\infty)$ up to a fixed time t. Set

$$T_a(t)=\int_0^t \mathbb{1}_{[a,+\infty)}(X(s))\,\mathrm{d}s.$$

Actually, this continuous-time functional is not well-defined since the pseudo-process X can be simultaneously defined only at a finite number of instants. Nevertheless, some ad-hoc definitions can be given, especially for computing certain expectations related to $T_a(t)$ (see definitions (1.2) and (1.3)).

The functional $T_a(t)$ has often been of interest: in [10], Krylov explicitly computed the distribution of $T_0(t)$ in the case where N is even and the starting point is exactly 0; he obtained the famous Lévy's arcsine law. In [9] and [18], Hochberg, Nikitin and Orsingher treated the case where N is equal to 3, 5 or 7 with a possible conditioning on an event depending on $X(t)$. In [11], Lachal explicitly determined the distribution of $T_0(t)$ in the general case (for any positive integer N) which is a Beta law. In a recent work [5], we have obtained a representation for the joint pseudo-distribution of the vector $(T_0(t),X(t))$ when the starting point is exactly $x=0$. For this, we introduced a construction of the pseudo-process based on observations of X on dyadic times and we used Spitzer's identity which works especially in the case where $x=0$.

The aim of this paper is to compute the joint pseudo-distribution of the vector $(T_a(t),X(t))$ for any starting point $x\in\mathbb{R}$, that is the pseudo-probability distribution function (ppdf in short)

$$\mathbb{P}_x\{T_a(t)\in \mathrm{d}s, X(t)\in \mathrm{d}y\}/(\mathrm{d}s\,\mathrm{d}y).$$

Since the pseudo-process X is invariant by translation, we have

$$\mathbb{P}_x\{T_a(t)\in \mathrm{d}s, X(t)\in \mathrm{d}y\}/(\mathrm{d}s\,\mathrm{d}y)=\mathbb{P}_{x-a}\{T_0(t)\in \mathrm{d}s, X(t)\in \mathrm{d}y-a\}/(\mathrm{d}s\,\mathrm{d}y)$$

and we only need to compute the ppdf $\mathbb{P}_x\{T(t)\in \mathrm{d}s, X(t)\in \mathrm{d}y\}/(\mathrm{d}s\,\mathrm{d}y)$, $s\in[0,t]$, $y\in\mathbb{R}$, where $T(t)=T_0(t)$. The approach we used in [5] is not efficient in the present situation since Spitzer's identity does not allow us to treat the general case of any starting point x different from 0. So, we follow here the Feynman-Kac approach which leads to solving partial differential equations. Set

$$\begin{aligned}
\chi(s,t;x,y) &= \mathbb{P}_x\{T(t)\in \mathrm{d}s, X(t)\in \mathrm{d}y\}/(\mathrm{d}s\,\mathrm{d}y),\\
\varpi_{\mu}(t;x,y) &= \mathbb{E}_x(\mathrm{e}^{-\mu T(t)}, X(t)\in \mathrm{d}y)/\mathrm{d}y = \int_0^t \mathrm{e}^{-\mu s}\chi(s,t;x,y)\,\mathrm{d}s,\\
\psi_{\lambda}(s;x,y) &= \int_s^{\infty} \mathrm{e}^{-\lambda t}\,\chi(s,t;x,y)\,\mathrm{d}t,\\
\varphi_{\lambda,\mu}(x,y) &= \int_0^{\infty} \mathrm{e}^{-\lambda t}\,\varpi_{\mu}(t;x,y)\,\mathrm{d}t,\\
\varrho_{\lambda}(x,y) &= \int_0^{\infty} \mathrm{e}^{-\lambda t}\,p(t;x-y)\,\mathrm{d}t.
\end{aligned}$$

Because of the lack of complete definition of the pseudo-process X over all continuous times, ad-hoc definitions for defining properly general functionals of X have been proposed. In this paper, it is enough to give only those concerning the functions χ, ϖ_{μ} and $\varphi_{\lambda,\mu}$; we refer the reader to [20] for more general functionals. The following definitions can be found in the literature: either

$$\begin{aligned}
\chi(s,t;x,y) &\overset{\text{def1}}{=} \lim_{n\to\infty} \mathbb{P}_x\{T_n^1(t)\in \mathrm{d}s, X(t)\in \mathrm{d}y\}/(\mathrm{d}s\,\mathrm{d}y),\\
\varpi_{\mu}(t;x,y) &\overset{\text{def1}}{=} \lim_{n\to\infty} \mathbb{E}_x\big(\mathrm{e}^{-\mu T_n^1(t)}, X(t)\in \mathrm{d}y\big)/\mathrm{d}y,\\
\varphi_{\lambda,\mu}(x,y) &\overset{\text{def1}}{=} \lim_{n\to\infty} \int_0^{\infty} \mathrm{e}^{-\lambda t}\big[\mathbb{E}_x\big(\mathrm{e}^{-\mu T_n^1(t)}, X(t)\in \mathrm{d}y\big)/\mathrm{d}y\big]\mathrm{d}t,
\end{aligned} \tag{1.2}$$

or

$$\begin{aligned}
\chi(s,t;x,y) &\overset{\text{def2}}{=} \lim_{n\to\infty} \mathbb{P}_x\{T_n^2(t)\in \mathrm{d}s, X([2^n t]/2^n)\in \mathrm{d}y\}/(\mathrm{d}s\,\mathrm{d}y),\\
\varpi_{\mu}(t;x,y) &\overset{\text{def2}}{=} \lim_{n\to\infty} \mathbb{E}_x\big(\mathrm{e}^{-\mu T_n^2(t)}, X([2^n t]/2^n)\in \mathrm{d}y\big)/\mathrm{d}y,\\
\varphi_{\lambda,\mu}(x,y) &\overset{\text{def2}}{=} \lim_{n\to\infty} \int_0^{\infty} \mathrm{e}^{-\lambda t}\big[\mathbb{E}_x\big(\mathrm{e}^{-\mu T_n^2(t)}, X([2^n t]/2^n)\in \mathrm{d}y\big)/\mathrm{d}y\big]\mathrm{d}t,
\end{aligned} \tag{1.3}$$

where

$$T_n^1(t) = \frac{t}{n}\sum_{k=0}^{n-1} 1\!\!1_{[0,+\infty)}(X(kt/n)), \quad T_n^2(t) = \frac{1}{2^n}\sum_{k=0}^{[2^n t]} 1\!\!1_{[0,+\infty)}(X(k/2^n)).$$

These definitions supply an appropriate support for computing the pseudo-distribution of $T(t)$. The first is particularly suitable for the Feynman-Kac approach (it is based on a subdivision of the interval $[0,t]$) while the second is convenient for the Spitzer approach (it is related to a subdivision of the time axis independent of t).

As in [10] for instance, the function ϖ_{μ} is a solution of

$$\kappa_N \frac{\partial^N \varpi_{\mu}}{\partial x^N}(t;x,y) = \frac{\partial \varpi_{\mu}}{\partial t}(t;x,y) + \mu 1\!\!1_{[0,+\infty)}(x)\varpi_{\mu}(t;x,y) \tag{1.4}$$

with initial condition $\varpi_{\mu}(0;x,y)=\delta_y(x)$ and the function $\varphi_{\lambda,\mu}$ is a Feynman-Kac functional which solves the differential equation

$$\kappa_N \frac{\partial^N \varphi_{\lambda,\mu}}{\partial x^N}(x,y) = \begin{cases} (\lambda+\mu)\,\varphi_{\lambda,\mu}(x,y)-\delta_y(x) & \text{if } x>0,\\ \lambda\,\varphi_{\lambda,\mu}(x,y)-\delta_y(x) & \text{if } x<0. \end{cases} \tag{1.5}$$

Together with (1.5), the function $\varphi_{\lambda,\mu}$ fulfills the conditions

$$\begin{cases} \varphi_{\lambda,\mu} \text{ is } (N-1) \text{ times differentiable at 0 and } (N-2) \text{ times differentiable at } y,\\ \dfrac{\partial^{N-1}\varphi_{\lambda,\mu}}{\partial x^{N-1}}(y^+,y) - \dfrac{\partial^{N-1}\varphi_{\lambda,\mu}}{\partial x^{N-1}}(y^-,y) = -\kappa_N. \end{cases} \tag{1.6}$$

An explanation of (1.4), (1.5) and (1.6), relies on an underlying integral equation which can be stated as follows. The function ϖ_{μ} satisfies the following integral equation:

$$\varpi_{\mu}(t;x,y) = p(t;x-y) - \mu\int_0^t\!\int_0^{+\infty} p(s;x-z)\,\varpi_{\mu}(t-s;z,y)\,\mathrm{d}s\,\mathrm{d}z, \tag{1.7}$$

and then, by taking the Laplace transform with respect to t, the function $\varphi_{\lambda,\mu}$ satisfies the integral equation below:

$$\varphi_{\lambda,\mu}(x,y) = \varrho_\lambda(x,y) - \mu \int_0^\infty \varrho_\lambda(x,z)\, \varphi_{\lambda,\mu}(z,y)\, \mathrm{d}z. \tag{1.8}$$

On one hand, since the heat-kernel p satisfies $\kappa_N \frac{\partial^N p}{\partial x^N}(t; x-y) = \frac{\partial p}{\partial t}(t; x-y)$ with $p(0; x-y) = \delta_y(x)$, we can see by differentiating with respect to t and x, that (1.7) yields the differential equation (1.4). On the other hand, since the function ϱ_λ satisfies $\kappa_N \frac{\partial^N \varrho_\lambda}{\partial x^N}(x,y) = \lambda\, \varrho_\lambda(x,y) - \delta_y(x)$ as well as conditions (1.6) (see [11]), we can see in the same manner that (1.8) yields the differential equation (1.5) together with conditions (1.6). An heuristic derivation of (1.7) consists in writing

$$\begin{aligned}
1 - \mathrm{e}^{-\mu T(t)} &= \mu \int_0^t 1\!\!1_{[0,+\infty)}(X(s)) \exp\left(-\mu \int_s^t 1\!\!1_{[0,+\infty)}(X(u))\, \mathrm{d}u\right) \mathrm{d}s \\
&= \mu \int_0^t 1\!\!1_{[0,+\infty)}(X(s))\, \mathrm{e}^{-\mu[T(t)-T(s)]}\, \mathrm{d}s = \mu \int_0^t 1\!\!1_{[0,+\infty)}(X(s))\, \mathrm{e}^{-\mu[T(t-s)\circ\theta_s]}\, \mathrm{d}s,
\end{aligned}$$

where $(\theta_s)_{s\geq 0}$ is the usual shift operator defined by $X(t)\circ\theta_s = X(s+t)$ for all $s,t \geq 0$, and next in applying the Markov property of the pseudo-process X:

$$\begin{aligned}
p(t; x-y) - \varpi_\mu(t; x, y) &= \mathbb{E}_x(1 - \mathrm{e}^{-\mu T(t)}, X(t) \in \mathrm{d}y)/\mathrm{d}y \\
&= \mu \int_0^t \left[\mathbb{E}_x\Big(1\!\!1_{[0,+\infty)}(X(s))\, \mathrm{e}^{-\mu[T(t-s)\circ\theta_s]}, X(t-s)\circ\theta_s \in \mathrm{d}y\Big) /\mathrm{d}y\right] \mathrm{d}s \\
&= \mu \int_0^t \int_0^{+\infty} \mathbb{P}_x\{X(s) \in \mathrm{d}z\} \left[\mathbb{E}_z\Big(\mathrm{e}^{-\mu T(t-s)}, X(t-s) \in \mathrm{d}y\Big) /\mathrm{d}y\right] \mathrm{d}s \\
&= \mu \int_0^t \int_0^{+\infty} p(s; x-z)\, \varpi_\mu(t-s; z, y)\, \mathrm{d}s\, \mathrm{d}z.
\end{aligned}$$

All this should be made rigorous (in the case where N is even at least) by introducing the step process obtained by sampling X on the dyadic times $k/2^n$, $k,n \in \mathbb{N}$, as, e.g., in [13] and [20]. We shall not go further in this direction. In this work instead, our aim is to solve the boundary value problem (1.5)-(1.6) and next to exhibit an explicit representation of the joint ppdf of the vector $(T_0(t), X(t))$ under $\mathbb{P}_x$.

The paper is organized as follows. Section 2 contains some settings. In Section 3 we completely solve the system (1.5)-(1.6) and derive explicitly $\varphi_{\lambda,\mu}(x,y)$. Section 4 and 5 are devoted to inverting the two-parameters Laplace transform $\varphi_{\lambda,\mu}(x,y)$. Our main result is displayed in Theorems 5.1 and 5.2. In each section, we shall split our analysis into two parts corresponding to the case where $y \geq 0$ and $y \leq 0$. Because of the asymmetry of the problem, we shall write out all the results associated with the aforementioned cases even if they look like similar. We shall provide all the details when $y \geq 0$ and omit the analogous computations related to the case where $y \leq 0$.

2 Settings

In order to solve Eq. (1.5) it is useful to introduce the N^{th} roots $(\theta_i)_{i\in I}$ of κ_N: $\theta_i^N = \kappa_N$ for all $i \in I$ (I contains N successive integers). We also introduce the sets of indices $J = \{i \in I : \mathrm{Re}(\theta_i) > 0\}$ and $K = \{i \in I : \mathrm{Re}(\theta_i) < 0\}$. We have $J \cup K = I$, $\#J + \#K = N$, $\#J = \#K = N/2$ if N is even, $|\#J - \#K| = 1$ if N is odd, $\kappa_N = (-1)^{\#J-1}$ and

$$\prod_{i\in I}(x - \theta_i) = x^N - \kappa_N. \tag{2.1}$$

Set, for $j, j' \in J$ and $k, k' \in K$,

$$A_j = \prod_{j'\in J\setminus\{j\}} \frac{\theta_{j'}}{\theta_{j'} - \theta_j}, \quad B_k = \prod_{k'\in K\setminus\{k\}} \frac{\theta_{k'}}{\theta_{k'} - \theta_k},$$

and

$$C_{jj'k} = \prod_{j''\in J} (\theta_j\theta_{j'} - \theta_{j''}\theta_k), \quad D_{jkk'} = \prod_{k''\in K} (\theta_k\theta_{k'} - \theta_{k''}\theta_j).$$

For $x=0$, formula (2.1) gives $\prod_{i\in I}\theta_i=(-1)^{N-1}\kappa_{_N}=(-1)^{\#K}$. On the other hand, since the non-real roots labeled by J (respectively by K) are conjugate two by two, we have

$$\prod_{j\in J}\theta_j=1 \quad\text{and}\quad \prod_{k\in K}\theta_k=(-1)^{\#K}. \tag{2.2}$$

For a fixed index $i\in I$, we have

$$\prod_{i'\in I\setminus\{i\}}(x-\theta_{i'})=\frac{x^N-\kappa_{_N}}{x-\theta_i}=\frac{x^N-\theta_i^N}{x-\theta_i}=\sum_{p=0}^{N-1}\theta_i^{N-1-p}x^p=\kappa_{_N}\sum_{p=0}^{N-1}\frac{x^p}{\theta_i^{p+1}},$$

which yields for $x=\theta_i$

$$\prod_{i'\in I\setminus\{i\}}(\theta_i-\theta_{i'})=\frac{\kappa_{_N}N}{\theta_i}. \tag{2.3}$$

By (2.2), we have for $j\in J$

$$\prod_{j'\in J\setminus\{j\}}(\theta_j-\theta_{j'})=(-1)^{\#J-1}\left(\prod_{j'\in J\setminus\{j\}}\theta_{j'}\right)\left(\prod_{j'\in J\setminus\{j\}}\frac{\theta_{j'}-\theta_j}{\theta_{j'}}\right)=\frac{\kappa_{_N}}{\theta_jA_j} \tag{2.4}$$

and for $k\in K$

$$\prod_{k'\in K\setminus\{k\}}(\theta_k-\theta_{k'})=(-1)^{\#K-1}\left(\prod_{k'\in K\setminus\{k\}}\theta_{k'}\right)\left(\prod_{k'\in K\setminus\{k\}}\frac{\theta_{k'}-\theta_k}{\theta_{k'}}\right)=-\frac{1}{\theta_kB_k}. \tag{2.5}$$

In view of (2.3) and (2.4), we get for $j'\in J$ and $k\in K$

$$\prod_{k'\in K}(\theta_{j'}-\theta_{k'})=\frac{\prod_{i\in I\setminus\{j'\}}(\theta_{j'}-\theta_i)}{\prod_{j''\in J\setminus\{j'\}}(\theta_{j'}-\theta_{j''})}=NA_{j'} \tag{2.6}$$

and next

$$\prod_{k'\in K\setminus\{k\}}(\theta_{j'}-\theta_{k'})=\frac{\prod_{k'\in K}(\theta_{j'}-\theta_{k'})}{\theta_{j'}-\theta_k}=\frac{NA_{j'}}{\theta_{j'}-\theta_k}. \tag{2.7}$$

We also recall the expression of the λ-potential $\varrho_{_\lambda}$ (see [11]):

$$\varrho_{_\lambda}(x,y)=\begin{cases}\dfrac{1}{N\lambda^{1-\frac1N}}\displaystyle\sum_{j\in J}\theta_j\mathrm{e}^{\theta_j\sqrt[N]{\lambda}\,(x-y)} & \text{for } x\in(-\infty,y],\\ -\dfrac{1}{N\lambda^{1-\frac1N}}\displaystyle\sum_{k\in K}\theta_k\mathrm{e}^{\theta_k\sqrt[N]{\lambda}\,(x-y)} & \text{for } x\in[y,+\infty).\end{cases} \tag{2.8}$$

3 Solving the system (1.5)-(1.6)

In this part, we explicitly solve the boundary problem (1.5)-(1.6). We divide our analysis into two cases: the case where $y\ge 0$ and the case where $y\le 0$.

3.1 Case $y\ge 0$

Assume that $y\ge 0$. Eq. (1.5) can be written as

$$\kappa_{_N}\frac{\partial^N\varphi_{_{\lambda,\mu}}}{\partial x^N}(x,y)=\begin{cases}\lambda\,\varphi_{_{\lambda,\mu}}(x,y) & \text{if } x\in(-\infty,0),\\ (\lambda+\mu)\,\varphi_{_{\lambda,\mu}}(x,y) & \text{if } x\in(0,y),\\ (\lambda+\mu)\,\varphi_{_{\lambda,\mu}}(x,y) & \text{if } x\in(y,+\infty).\end{cases}$$

In the particular case where $y=0$, the interval $(0,y)$ is empty. The solution of this equation has the form

$$\varphi_{_{\lambda,\mu}}(x,y)=\begin{cases}\displaystyle\sum_{i\in I}b_i(y)\,\mathrm{e}^{\theta_i\sqrt[N]{\lambda}\,x} & \text{if } x\in(-\infty,0),\\ \displaystyle\sum_{i\in I}a_i(y)\,\mathrm{e}^{\theta_i\sqrt[N]{\lambda+\mu}\,x} & \text{if } x\in(0,y),\\ \displaystyle\sum_{i\in I}c_i(y)\,\mathrm{e}^{\theta_i\sqrt[N]{\lambda+\mu}\,x} & \text{if } x\in(y,+\infty),\end{cases}$$

where the unknowns $(a_i(y))_{i\in I}$, $(b_i(y))_{i\in I}$ and $(c_i(y))_{i\in I}$ are to be determined.

First, by combining (1.8) and (2.8), we see that, for large enough negative x, $\varphi_{\lambda,\mu}(x,y)$ is a linear combination of $\mathrm{e}^{\theta_j\sqrt[N]{\lambda}\,x}$, $j\in J$. Then $b_i=0$ for all $i\in K$. Second, by (1.8), we see that the integral $\int_0^\infty \varrho_\lambda(x,z)\,\varphi_{\lambda,\mu}(z,y)\,\mathrm{d}z$ must be convergent. Due to (2.8), this implies that for large enough positive x, $\varphi_{\lambda,\mu}(x,y)$ is only a linear combination of $\mathrm{e}^{\theta_k\sqrt[N]{\lambda+\mu}\,x}$, $k\in K$. Thus, $c_i=0$ for any $i\in J$. As a byproduct, we have to search $\varphi_{\lambda,\mu}$ in the form

$$\varphi_{\lambda,\mu}(x,y)=\begin{cases}\displaystyle\sum_{j\in J} b_j(y)\,\mathrm{e}^{\theta_j\sqrt[N]{\lambda}\,x} & \text{if } x\in(-\infty,0),\\ \displaystyle\sum_{i\in I} a_i(y)\,\mathrm{e}^{\theta_i\sqrt[N]{\lambda+\mu}\,x} & \text{if } x\in(0,y),\\ \displaystyle\sum_{k\in K} c_k(y)\,\mathrm{e}^{\theta_k\sqrt[N]{\lambda+\mu}\,x} & \text{if } x\in(y,+\infty).\end{cases}$$

By the conditions (1.6), the unknowns $(a_i(y))_{i\in I}$, $(b_j(y))_{j\in J}$ and $(c_k(y))_{k\in K}$ verify

$$\begin{cases}\displaystyle\lambda^{\frac{p}{N}}\sum_{j\in J} b_j(y)\theta_j^p=(\lambda+\mu)^{\frac{p}{N}}\sum_{i\in I} a_i(y)\theta_i^p & \text{if } 0\le p\le N-1,\\ \displaystyle\sum_{i\in I} a_i(y)\theta_i^p\mathrm{e}^{\theta_i\sqrt[N]{\lambda+\mu}\,y}-\sum_{k\in K} c_k(y)\theta_k^p\mathrm{e}^{\theta_k\sqrt[N]{\lambda+\mu}\,y}=\begin{cases}0\\ \dfrac{\kappa_N}{(\lambda+\mu)^{1-1/N}}\end{cases} & \begin{array}{l}\text{if } 0\le p\le N-2,\\ \text{if } p=N-1.\end{array}\end{cases}$$

In order to simplify the notations we set $\gamma=\sqrt[N]{\lambda+\mu}$ and $\delta=\sqrt[N]{\lambda}$. The system writes

$$\begin{cases}\displaystyle\delta^p\sum_{j\in J} b_j(y)\theta_j^p=\gamma^p\sum_{i\in I} a_i(y)\theta_i^p & \text{if } 0\le p\le N-1,\\ \displaystyle\sum_{j\in J} a_j(y)\theta_j^p\mathrm{e}^{\theta_j\gamma y}+\sum_{k\in K}(a_k(y)-c_k(y))\theta_k^p\mathrm{e}^{\theta_k\gamma y}=\begin{cases}0\\ \dfrac{\kappa_N}{\gamma^{N-1}}\end{cases} & \begin{array}{l}\text{if } 0\le p\le N-2,\\ \text{if } p=N-1.\end{array}\end{cases}\tag{3.1}$$

Put

$$\tilde{a}_i(y)=\begin{cases}a_i(y)\,\mathrm{e}^{\theta_i\gamma y} & \text{if } i\in J,\\ (a_i(y)-c_i(y))\,\mathrm{e}^{\theta_i\gamma y} & \text{if } i\in K.\end{cases}$$

The second equation of (3.1) yields the following system of N equations with the N unknowns $\tilde{a}_i(y)$, $i\in I$:

$$\sum_{i\in I}\tilde{a}_i(y)\theta_i^p=\begin{cases}0 & \text{if } 0\le p\le N-2,\\ \dfrac{\kappa_N}{\gamma^{N-1}} & \text{if } p=N-1.\end{cases}$$

This is a classical Vandermonde system the solution of which is

$$\tilde{a}_i(y)=\frac{\kappa_N}{\gamma^{N-1}\prod_{i'\in I\setminus\{i\}}(\theta_i-\theta_{i'})},\quad i\in I.$$

In view of (2.3) we have

$$\tilde{a}_i(y)=\frac{\theta_i}{N\gamma^{N-1}},\quad i\in I.$$

We notice that $\tilde{a}_i(y)$ does not depend on y. We deduce that

$$\begin{cases}a_j(y)=\dfrac{\theta_j}{N\gamma^{N-1}}\,\mathrm{e}^{-\theta_j\gamma y} & \text{if } j\in J,\\[2ex] a_k(y)=\dfrac{\theta_k}{N\gamma^{N-1}}\,\mathrm{e}^{-\theta_k\gamma y}+c_k(y) & \text{if } k\in K.\end{cases}$$

Now, we need to compute the $c_k(y)$, $k\in K$. The first equation of (3.1) yields

$$\sum_{j\in J} b_j(y)(\theta_j\delta)^p+\sum_{k\in K}(-a_k(y))(\theta_k\gamma)^p=\sum_{j\in J} a_j(y)(\theta_j\gamma)^p\quad\text{for } 0\le p\le N-1.$$

This can be rewritten into a matrix form as

$$\left(\left((\theta_j\delta)^p\right)_{\substack{0\le p\le N-1\\ j\in J}} \;\vdots\; \left((\theta_k\gamma)^p\right)_{\substack{0\le p\le N-1\\ k\in K}}\right)\times\begin{pmatrix}(b_j(y))_{j\in J}\\ (-a_k(y))_{k\in K}\end{pmatrix}=\sum_{j\in J}a_j(y)\left(\left((\theta_j\gamma)^p\right)_{0\le p\le N-1}\right) \tag{3.2}$$

where the index $p\in\{0,\dots,N-1\}$ in $(\theta_j\delta)^p$ and $(\theta_k\gamma)^p$ stands for the row index and the indices $j\in J$ in $(\theta_j\delta)^p$ and $k\in K$ in $(\theta_k\gamma)^p$ stand for the column indices. The quantities $(b_j(y))_{j\in J}$, $(-a_k(y))_{k\in K}$ and $((\theta_j\gamma)^p)_{0\le p\le N-1}$ are 1-column matrices.

Put $\vartheta_j=\theta_j\delta$ for $j\in J$ and $\vartheta_k=\theta_k\gamma$ for $k\in K$, $\alpha_j(y)=b_j(y)$ for $j\in J$ and $\alpha_k(y)=-a_k(y)$ for $k\in K$. From (3.2) we obtain the following Vandermonde system:

$$\left((\vartheta_i^p)_{\substack{0\le p\le N-1\\ i\in I}}\right)\times\left((\alpha_i(y))_{i\in I}\right)=\sum_{j\in J}a_j(y)\left(((\theta_j\gamma)^p)_{0\le p\le N-1}\right). \tag{3.3}$$

In order to solve (3.3), we first solve, for each $j\in J$, the partial Vandermonde system below:

$$\sum_{i\in I}\alpha_i\vartheta_i^p=(\theta_j\gamma)^p,\quad 0\le p\le N-1. \tag{3.4}$$

The solution of (3.4) is given by

$$\alpha_i=\prod_{i'\in I\setminus\{i\}}\frac{\theta_j\gamma-\vartheta_{i'}}{\vartheta_i-\vartheta_{i'}},\quad i\in I. \tag{3.5}$$

Indeed, the polynomial $\mathcal{A}$ defined by

$$\mathcal{A}(x)=\sum_{i\in I}\vartheta_i^p\prod_{i'\in I\setminus\{i\}}\frac{x-\vartheta_{i'}}{\vartheta_i-\vartheta_{i'}}$$

is such that $\mathcal{A}(\vartheta_i)=\vartheta_i^p$, $i\in I$ and degree$(\mathcal{A})\le\#I-1=N-1$. The polynomial x^p satisfies the same conditions. Since these conditions uniquely determine the polynomial $\mathcal{A}$ (this is the so-called Lagrange interpolating polynomial), we deduce that $\mathcal{A}(x)=x^p$ for all x. Thus, choosing α_i as in (3.5), $\mathcal{A}(\theta_j\gamma)=\sum_{i\in I}\alpha_i\vartheta_i^p=(\theta_j\gamma)^p$ which proves that (3.5) is the solution of (3.4).

Now the solution of the system (3.3) is obtained by taking the linear combination of the foregoing solutions (3.5):

$$\alpha_i(y)=\sum_{j'\in J}a_{j'}(y)\prod_{i'\in I\setminus\{i\}}\frac{\theta_{j'}\gamma-\vartheta_{i'}}{\vartheta_i-\vartheta_{i'}},\quad i\in I. \tag{3.6}$$

Recalling that $a_{j'}(y)=\frac{\theta_{j'}}{N\gamma^{N-1}}\,\mathrm{e}^{-\theta_{j'}\gamma y}$ for $j'\in J$, (3.6) yields the following expressions for $b_j(y)$, $j\in J$, and $a_k(y)$, $c_k(y)$, $k\in K$.

- For $j\in J$, we have

$$b_j(y)=\frac{1}{N\gamma^{N-1}}\sum_{j'\in J}\theta_{j'}\left(\prod_{i'\in I\setminus\{j\}}\frac{\theta_{j'}\gamma-\vartheta_{i'}}{\vartheta_j-\vartheta_{i'}}\right)\mathrm{e}^{-\theta_{j'}\gamma y}$$

where

$$\begin{aligned}\prod_{i'\in I\setminus\{j\}}\frac{\theta_{j'}\gamma-\vartheta_{i'}}{\vartheta_j-\vartheta_{i'}}&=\prod_{j''\in J\setminus\{j\}}\frac{\theta_{j'}\gamma-\theta_{j''}\delta}{(\theta_j-\theta_{j''})\delta}\times\prod_{k\in K}\frac{(\theta_{j'}-\theta_k)\gamma}{\theta_j\delta-\theta_k\gamma}\\&=\frac{\gamma^{\#K}}{\delta^{\#J-1}}\,\frac{\prod_{k\in K}(\theta_{j'}-\theta_k)}{\prod_{j''\in J\setminus\{j\}}(\theta_j-\theta_{j''})}\times\frac{\prod_{j''\in J\setminus\{j\}}(\theta_{j'}\gamma-\theta_{j''}\delta)}{\prod_{k\in K}(\theta_j\delta-\theta_k\gamma)}.\end{aligned}$$

In view of (2.4) and (2.6), we see that

$$\frac{\prod_{k\in K}(\theta_{j'}-\theta_k)}{\prod_{j''\in J\setminus\{j\}}(\theta_j-\theta_{j''})}=\kappa_{N}N\theta_jA_jA_{j'}.$$

Therefore

$$b_j(y)=\frac{\kappa_{N}\theta_jA_j}{(\gamma\delta)^{\#J-1}}\sum_{j'\in J}\theta_{j'}A_{j'}\left(\frac{\prod_{j''\in J\setminus\{j\}}(\theta_{j'}\gamma-\theta_{j''}\delta)}{\prod_{k\in K}(\theta_j\delta-\theta_k\gamma)}\right)\mathrm{e}^{-\theta_{j'}\gamma y},\quad j\in J.$$

- For $k \in K$, we have

$$a_k(y) = -\frac{1}{N\gamma^{N-1}} \sum_{j' \in J} \theta_{j'} \left(\prod_{i' \in I \setminus \{k\}} \frac{\theta_{j'}\gamma - \vartheta_{i'}}{\vartheta_k - \vartheta_{i'}} \right) \mathrm{e}^{-\theta_{j'}\gamma y}$$

where

$$\prod_{i' \in I \setminus \{k\}} \frac{\theta_{j'}\gamma - \vartheta_{i'}}{\vartheta_k - \vartheta_{i'}} = \prod_{j \in J} \frac{\theta_{j'}\gamma - \theta_j \delta}{\theta_k \gamma - \theta_j \delta} \times \prod_{k' \in K \setminus \{k\}} \frac{\theta_{j'} - \theta_{k'}}{\theta_k - \theta_{k'}}.$$

In view of (2.5) and (2.7), we see that

$$\prod_{k' \in K \setminus \{k\}} \frac{\theta_{j'} - \theta_{k'}}{\theta_k - \theta_{k'}} = -N \frac{A_{j'}\theta_k B_k}{\theta_{j'} - \theta_k}.$$

Therefore

$$a_k(y) = \frac{\theta_k B_k}{\gamma^{N-1}} \sum_{j' \in J} \frac{\theta_{j'} A_{j'}}{\theta_{j'} - \theta_k} \left(\prod_{j \in J} \frac{\theta_{j'}\gamma - \theta_j \delta}{\theta_k \gamma - \theta_j \delta} \right) \mathrm{e}^{-\theta_{j'}\gamma y}, \quad k \in K.$$

We finally obtain that

$$c_k(y) = a_k(y) - \frac{\theta_k}{N\gamma^{N-1}} \mathrm{e}^{-\theta_k \gamma y}, \quad k \in K.$$

As a byproduct we have obtained the form below for the function $\varphi_{\lambda,\mu}(x, y)$.

Proposition 3.1 *Suppose $y \geq 0$. The function $\varphi_{\lambda,\mu}(x, y)$ admits the following representation:*

for $x \in (-\infty, 0]$,

$$\varphi_{\lambda,\mu}(x, y) = \frac{\kappa_N}{(\gamma\delta)^{\#J-1}} \sum_{j,j' \in J} \theta_j A_j \theta_{j'} A_{j'} \frac{\prod_{j'' \in J \setminus \{j\}} (\theta_{j'}\gamma - \theta_{j''}\delta)}{\prod_{k \in K} (\theta_j \delta - \theta_k \gamma)} \mathrm{e}^{\theta_j \delta x - \theta_{j'}\gamma y};$$

for $x \in [0, y]$,

$$\varphi_{\lambda,\mu}(x, y) = \frac{1}{\gamma^{N-1}} \left[\frac{1}{N} \sum_{j \in J} \theta_j \mathrm{e}^{\theta_j \gamma (x-y)} + \sum_{j \in J, k \in K} \frac{\theta_j A_j \theta_k B_k}{\theta_j - \theta_k} \prod_{j' \in J} \left(\frac{\theta_j \gamma - \theta_{j'}\delta}{\theta_k \gamma - \theta_{j'}\delta} \right) \mathrm{e}^{\gamma(\theta_k x - \theta_j y)} \right];$$

for $x \in [y, +\infty)$,

$$\varphi_{\lambda,\mu}(x, y) = \frac{1}{\gamma^{N-1}} \left[-\frac{1}{N} \sum_{k \in K} \theta_k \mathrm{e}^{\theta_k \gamma (x-y)} + \sum_{j \in J, k \in K} \frac{\theta_j A_j \theta_k B_k}{\theta_j - \theta_k} \prod_{j' \in J} \left(\frac{\theta_j \gamma - \theta_{j'}\delta}{\theta_k \gamma - \theta_{j'}\delta} \right) \mathrm{e}^{\gamma(\theta_k x - \theta_j y)} \right].$$

Remark 3.1 *If we take $\mu = 0$ in $\varphi_{\lambda,\mu}(x, y)$, we retrieve the λ-potential of the pseudo-process X: $\varphi_{\lambda,\mu}(x, y) = \varrho_\lambda(x, y)$. Explicitly, we have $\gamma = \delta = \sqrt[N]{\lambda}$,*

$$\prod_{j' \in J} \left(\frac{\theta_j \gamma - \theta_{j'}\delta}{\theta_k \gamma - \theta_{j'}\delta} \right) = \prod_{j' \in J} \left(\frac{\theta_j - \theta_{j'}}{\theta_k - \theta_{j'}} \right) = 0,$$

and by (2.4) and (2.6)

$$\frac{\prod_{j'' \in J \setminus \{j\}} (\theta_{j'}\gamma - \theta_{j''}\delta)}{\prod_{k \in K} (\theta_j \delta - \theta_k \gamma)} = \delta^{\#J - \#K - 1} \frac{\prod_{j'' \in J \setminus \{j\}} (\theta_{j'} - \theta_{j''})}{\prod_{k \in K} (\theta_j - \theta_k)} = \delta_{jj'} \delta^{\#J - \#K - 1} \frac{\kappa_N}{N A_j \theta_{j'} A_{j'}}.$$

As a result, the formulas of Proposition 3.1 supply (2.8). We can sum up the two expressions of $\varphi_{\lambda,\mu}(x, y)$ for $x \in [0, y]$ and $x \in [y, +\infty)$ into the following one: for $x \in [0, +\infty)$,

$$\varphi_{\lambda,\mu}(x, y) = \varrho_{\lambda+\mu}(x, y) + \frac{1}{\gamma^{N-1}} \sum_{j \in J, k \in K} \frac{\theta_j A_j \theta_k B_k}{\theta_j - \theta_k} \prod_{j' \in J} \left(\frac{\theta_j \gamma - \theta_{j'}\delta}{\theta_k \gamma - \theta_{j'}\delta} \right) \mathrm{e}^{\gamma(\theta_k x - \theta_j y)}.$$

Remark 3.2 *Suppose that $y = 0$. By Proposition 3.1, we have on the one hand for $x \leq 0$*

$$\varphi_{\lambda,\mu}(x, 0) = \frac{\kappa_N}{(\gamma\delta)^{\#J-1}} \sum_{j \in J} \theta_j A_j \left[\sum_{j' \in J} \theta_{j'} A_{j'} \prod_{j'' \in J \setminus \{j\}} (\theta_{j'}\gamma - \theta_{j''}\delta) \right] \frac{\mathrm{e}^{\theta_j \delta x}}{\prod_{k \in K} (\theta_j \delta - \theta_k \gamma)}.$$

After several computations which are postponed to Appendix A, we get, for $x \le 0$,

$$\varphi_{\lambda,\mu}(x,0) = \int_0^\infty \mathrm{e}^{-\lambda t}[\mathbb{E}_x(\mathrm{e}^{-\mu T(t)}, X(t) \in \mathrm{d}y)/\mathrm{d}y]\Big|_{y=0}\mathrm{d}t$$
$$= \frac{\sqrt[N]{\lambda}}{\mu} \sum_{j\in J} \theta_j A_j \left[\prod_{j'\in J} \left(\theta_{j'} \sqrt[N]{\frac{\lambda+\mu}{\lambda}} - \theta_j \right) \right] \mathrm{e}^{\theta_j \sqrt[N]{\lambda}\, x}. \tag{3.7}$$

It is then easy to retrieve Formula (24) of [11]. Choosing now $x = 0$, $\varphi_{\lambda,\mu}(x,0)$ yields

$$\varphi_{\lambda,\mu}(0,0) = \frac{\sqrt[N]{\lambda}}{\mu} \sum_{j\in J} \theta_j A_j \left[\prod_{j'\in J} \left(\theta_{j'} \sqrt[N]{\frac{\lambda+\mu}{\lambda}} - \theta_j \right) \right].$$

We can see (cf. again Appendix A) that

$$\varphi_{\lambda,\mu}(0,0) = \int_0^\infty \mathrm{e}^{-\lambda t}[\mathbb{E}_0(\mathrm{e}^{-\mu T(t)}, X(t) \in \mathrm{d}y)/\mathrm{d}y]\Big|_{y=0}\mathrm{d}t = \left(\sum_{j\in J} \theta_j \right) \frac{\sqrt[N]{\lambda+\mu} - \sqrt[N]{\lambda}}{\mu}. \tag{3.8}$$

This is Formula (26) of [11] which leads to the famous uniform distribution of the sojourn time in $[0,+\infty)$ for the pseudo-bridge process $(X(s)|X(t)=0)_{0\le s\le t}$ (see Theorem 13 of [11]):

$$\mathbb{P}_0\{T(t) \in \mathrm{d}s | X(t) = 0\}/\mathrm{d}s = \frac{1}{t}\, \mathbb{1}_{[0,t]}(s).$$

On the other hand, we have for $x \ge 0$

$$\varphi_{\lambda,\mu}(x,0) = \frac{1}{\gamma^{N-1}} \sum_{k\in K} \theta_k B_k \left[\sum_{j\in J} \frac{\theta_j A_j}{\theta_j - \theta_k} \prod_{j'\in J} \left(\frac{\theta_j \gamma - \theta_{j'}\delta}{\theta_k \gamma - \theta_{j'}\delta} \right) - \frac{1}{NB_k} \right] \mathrm{e}^{\theta_k \gamma x}.$$

As previously, after several computations which are postponed to Appendix A, we get, for $x \ge 0$,

$$\varphi_{\lambda,\mu}(x,0) = \frac{\sqrt[N]{\lambda+\mu}}{\mu} \sum_{k\in K} \theta_k B_k \left[\prod_{k'\in K} \left(\theta_k - \theta_{k'} \sqrt[N]{\frac{\lambda}{\lambda+\mu}} \right) \right] \mathrm{e}^{\theta_k \sqrt[N]{\lambda+\mu}\, x}. \tag{3.9}$$

It is easy to retrieve Formula (24) of [11] in this case.

Remark 3.3 *Suppose that $x = 0$ and $y \ge 0$. Proposition 3.1 yields in this case*

$$\varphi_{\lambda,\mu}(0,y) = \frac{1}{\gamma^{N-1}} \sum_{j\in J} \theta_j A_j \left[\sum_{k\in K} \frac{\theta_k B_k}{\theta_j - \theta_k} \prod_{j'\in J} \left(\frac{\theta_j \gamma - \theta_{j'}\delta}{\theta_k \gamma - \theta_{j'}\delta} \right) + \frac{1}{NA_j} \right] \mathrm{e}^{-\theta_j \gamma y}.$$

In Appendix B, we show that $\varphi_{\lambda,\mu}(0,y)$ can be rewritten as

$$\varphi_{\lambda,\mu}(0,y) = -\frac{1}{\gamma^{\#J-1}\delta^{\#K-1}} \sum_{j\in J} \theta_j A_j \left(\sum_{k\in K} \frac{\theta_k B_k}{\theta_j \gamma - \theta_k \delta} \right) \mathrm{e}^{-\theta_j \gamma y}. \tag{3.10}$$

We retrieve Formula (4.1) of [5].

The expressions lying in Proposition 3.1 are not tractable for inverting the Laplace transform $\varphi_{\lambda,\mu}(x,y)$. So we transform them in order to derive a more appropriate form. For this we introduce the rational fractions defined, for $j, j' \in J$ and $k \in K$, as

$$F_{jj'}(x) = \frac{\prod_{j''\in J\setminus\{j\}} (\theta_{j'} x - \theta_{j''})}{\prod_{k\in K} (\theta_k x - \theta_j)}, \quad G_{jk}(x) = \prod_{j'\in J} \left(\frac{\theta_{j'} x - \theta_j}{\theta_{j'} x - \theta_k} \right).$$

Let us expand them into partial fractions.

- We first observe that

$$F_{jj'}(x)=\frac{\theta_{j'}^{\#J-1}}{\prod_{k\in K}\theta_k}1\!\!1_{\{\#J=\#K+1\}}+\sum_{k\in K}\frac{\alpha_{jj'k}}{\theta_k x-\theta_j}$$

with

$$\alpha_{jj'k}=\lim_{x\to\theta_j/\theta_k}(\theta_k x-\theta_j)F_{jj'}(x)=\frac{\prod_{j''\in J\setminus\{j\}}\left(\frac{\theta_j\theta_{j'}}{\theta_k}-\theta_{j''}\right)}{\prod_{k'\in K\setminus\{k\}}\left(\frac{\theta_j\theta_{k'}}{\theta_k}-\theta_j\right)}=\frac{\theta_k^{\#K-\#J}}{\theta_j^{\#K-1}}\,\frac{\prod_{j''\in J\setminus\{j\}}(\theta_j\theta_{j'}-\theta_{j''}\theta_k)}{\prod_{k'\in K\setminus\{k\}}(\theta_{k'}-\theta_k)}.$$

By (2.6) we extract

$$\alpha_{jj'k}=(-1)^{\#K}\frac{\theta_k^{\#K-\#J+1}B_kC_{jj'k}}{\theta_j^{\#K}(\theta_{j'}-\theta_k)}$$

and then

$$F_{jj'}(x)=(-1)^{\#K}\theta_{j'}^{\#K}1\!\!1_{\{\#J=\#K+1\}}+\frac{(-1)^{\#K}}{\theta_j^{\#K}}\sum_{k\in K}\frac{\theta_k^{\#K-\#J+1}B_kC_{jj'k}}{(\theta_{j'}-\theta_k)(\theta_k x-\theta_j)}.$$

- Analogously we observe that

$$G_{jk}(x)=1+\sum_{j'\in J}\frac{\beta_{jj'k}}{\theta_{j'}x-\theta_k}$$

with

$$\beta_{jj'k}=\lim_{x\to\theta_k/\theta_{j'}}(\theta_{j'}x-\theta_k)G_{jk}(x)=\frac{\prod_{j''\in J}\left(\frac{\theta_{j''}\theta_k}{\theta_{j'}}-\theta_j\right)}{\prod_{j''\in J\setminus\{j'\}}\left(\frac{\theta_{j''}\theta_k}{\theta_{j'}}-\theta_k\right)}=\frac{1}{\theta_{j'}\theta_k^{\#J-1}}\,\frac{\prod_{j''\in J}(\theta_{j''}\theta_k-\theta_j\theta_{j'})}{\prod_{j''\in J\setminus\{j'\}}(\theta_{j''}-\theta_{j'})}.$$

By (2.5) we extract

$$\beta_{jj'k}=(-1)^{\#J}\frac{A_{j'}C_{jj'k}}{\theta_k^{\#J-1}}$$

and then

$$G_{jk}(x)=1+\frac{(-1)^{\#J}}{\theta_k^{\#J-1}}\sum_{j'\in J}\frac{A_{j'}C_{jj'k}}{\theta_{j'}x-\theta_k}.$$

Therefore we can rewrite the fractions lying in Proposition 3.1 as

$$\begin{aligned}\frac{\prod_{j''\in J\setminus\{j\}}(\theta_{j'}\gamma-\theta_{j''}\delta)}{\prod_{k\in K}(\theta_j\delta-\theta_k\gamma)}&=(-1)^{\#K}\delta^{\#J-\#K-1}F_{jj'}(\gamma/\delta)\\&=\theta_{j'}^{\#J-1}1\!\!1_{\{\#J=\#K+1\}}+\frac{\delta^{\#J-\#K}}{\theta_j^{\#K}}\sum_{k\in K}\frac{\theta_k^{\#K-\#J+1}B_kC_{jj'k}}{(\theta_{j'}-\theta_k)(\theta_k\gamma-\theta_j\delta)}\end{aligned}\tag{3.11}$$

and

$$\prod_{j'\in J}\frac{\theta_j\gamma-\theta_{j'}\delta}{\theta_k\gamma-\theta_{j'}\delta}=G_{jk}(\delta/\gamma)=1+\frac{(-1)^{\#J-1}}{\theta_k^{\#J-1}}\,\gamma\sum_{j'\in J}\frac{A_{j'}C_{jj'k}}{\theta_k\gamma-\theta_{j'}\delta}.\tag{3.12}$$

With (3.11) and (3.12) at hands, we derive the new expression below for $\varphi_{\lambda,\mu}(x,y)$. Recall that $\gamma=\sqrt[N]{\lambda+\mu}$ and $\delta=\sqrt[N]{\lambda}$.

Proposition 3.2 *Suppose $y\ge 0$. The function $\varphi_{\lambda,\mu}(x,y)$ can be written as follows:*

for $x\in(-\infty,0]$,

$$\begin{aligned}\varphi_{\lambda,\mu}(x,y)=&\frac{\kappa_N\,1\!\!1_{\{\#J=\#K+1\}}}{(\lambda+\mu)^{\frac{\#K}{N}}\lambda^{\frac{\#K}{N}}}\left[\sum_{j\in J}\theta_jA_j\mathrm{e}^{\theta_j\sqrt[N]{\lambda}\,x}\right]\left[\sum_{j\in J}\theta_j^{\#J}A_j\mathrm{e}^{-\theta_j\sqrt[N]{\lambda+\mu}\,y}\right]\\&+\frac{\kappa_N}{(\lambda+\mu)^{\frac{\#J-1}{N}}\lambda^{\frac{\#K-1}{N}}}\sum_{j,j'\in J,\,k\in K}\frac{A_j\theta_{j'}A_{j'}\theta_k^{\#K-\#J+1}B_kC_{jj'k}}{\theta_j^{\#K-1}(\theta_{j'}-\theta_k)}\,\frac{\mathrm{e}^{\theta_j\sqrt[N]{\lambda}\,x-\theta_{j'}\sqrt[N]{\lambda+\mu}\,y}}{\theta_k\sqrt[N]{\lambda+\mu}-\theta_j\sqrt[N]{\lambda}};\end{aligned}$$

for $x\in[0,+\infty)$,

$$\varphi_{\lambda,\mu}(x,y)=\varrho_{\lambda+\mu}(x,y)+\frac{1}{(\lambda+\mu)^{1-\frac{1}{N}}}\left[\sum_{j\in J,\,k\in K}\frac{\theta_jA_j\theta_kB_k}{\theta_j-\theta_k}\,\mathrm{e}^{\sqrt[N]{\lambda+\mu}\,(\theta_kx-\theta_jy)}\right.$$

$$\left.+\kappa_N\sqrt[N]{\lambda+\mu}\sum_{j,j'\in J,\,k\in K}\frac{\theta_jA_jA_{j'}B_kC_{jj'k}}{\theta_k^{\#J-2}(\theta_j-\theta_k)}\,\frac{\mathrm{e}^{\sqrt[N]{\lambda+\mu}\,(\theta_kx-\theta_jy)}}{\theta_k\sqrt[N]{\lambda+\mu}-\theta_{j'}\sqrt[N]{\lambda}}\right].$$

3.2 Case $y\le 0$

Assume now that $y\le 0$. In this case Eq. (1.5) can be written as

$$\kappa_N\,\frac{\partial^N\varphi_{\lambda,\mu}}{\partial x^N}(x,y)=\begin{cases}\lambda\,\varphi_{\lambda,\mu}(x,y) & \text{if } x\in(-\infty,y),\\ \lambda\,\varphi_{\lambda,\mu}(x,y) & \text{if } x\in(y,0),\\ (\lambda+\mu)\,\varphi_{\lambda,\mu}(x,y) & \text{if } x\in(0,+\infty).\end{cases}$$

As in the foregoing case, the solution of this equation has the form

$$\varphi_{\lambda,\mu}(x,y)=\begin{cases}\displaystyle\sum_{j\in J}b_j(y)\,\mathrm{e}^{\theta_j\sqrt[N]{\lambda}\,x} & \text{if } x\in(-\infty,y),\\ \displaystyle\sum_{i\in I}a_i(y)\,\mathrm{e}^{\theta_i\sqrt[N]{\lambda}\,x} & \text{if } x\in(y,0),\\ \displaystyle\sum_{k\in K}c_k(y)\,\mathrm{e}^{\theta_k\sqrt[N]{\lambda+\mu}\,x} & \text{if } x\in(0,+\infty),\end{cases}$$

where the unknowns $(a_i(y))_{i\in I}$, $(b_j(y))_{j\in J}$ and $(c_k(y))_{k\in K}$ satisfy, due to (1.6),

$$\begin{cases}\displaystyle(\lambda+\mu)^{\frac{p}{N}}\sum_{k\in K}c_k(y)\theta_k^p=\lambda^{\frac{p}{N}}\sum_{i\in I}a_i(y)\theta_i^p & \text{if } 0\le p\le N-1,\\ \displaystyle\sum_{i\in I}a_i(y)\theta_i^p\mathrm{e}^{\theta_i\sqrt[N]{\lambda}\,y}-\sum_{j\in J}b_j(y)\theta_j^p\mathrm{e}^{\theta_j\sqrt[N]{\lambda}\,y}=\begin{cases}0 & \\ \dfrac{-\kappa_N}{\lambda^{1-1/N}} & \end{cases} & \begin{matrix}\text{if } 0\le p\le N-2,\\ \text{if } p=N-1.\end{matrix}\end{cases}$$

With calculations analogous to those performed in the case $y\ge 0$ we obtain the result below.

Proposition 3.3 *Suppose $y\le 0$. The function $\varphi_{\lambda,\mu}(x,y)$ admits the following representation:*

for $x\in(-\infty,0]$,

$$\varphi_{\lambda,\mu}(x,y)=\varrho_\lambda(x,y)-\frac{1}{\delta^{N-1}}\sum_{j\in J,\,k\in K}\frac{\theta_jA_j\theta_kB_k}{\theta_k-\theta_j}\prod_{k'\in K}\left(\frac{\theta_k\delta-\theta_{k'}\gamma}{\theta_j\delta-\theta_{k'}\gamma}\right)\mathrm{e}^{\delta(\theta_jx-\theta_ky)};$$

for $x\in[0,+\infty)$,

$$\varphi_{\lambda,\mu}(x,y)=\frac{-\kappa_N}{(\gamma\delta)^{\#K-1}}\sum_{k,k'\in K}\theta_kB_k\theta_{k'}B_{k'}\,\frac{\prod_{k''\in K\setminus\{k\}}(\theta_{k'}\delta-\theta_{k''}\gamma)}{\prod_{j\in J}(\theta_k\gamma-\theta_j\delta)}\,\mathrm{e}^{\theta_k\gamma x-\theta_{k'}\delta y}.$$

Remark 3.4 *Letting μ tend to $+\infty$ (and then $\gamma\to+\infty$), we have that $\frac{\theta_k\delta-\theta_{k'}\gamma}{\theta_j\delta-\theta_{k'}\gamma}\to 1$ and $\mathrm{e}^{\theta_k\gamma x}\to 0$ for $x>0$ and $k\in K$, and then*

$$\varphi_{\lambda,\mu}(x,y)\longrightarrow\begin{cases}\displaystyle\varrho_\lambda(x,y)-\frac{1}{\delta^{N-1}}\sum_{j\in J,\,k\in K}\frac{\theta_jA_j\theta_kB_k}{\theta_k-\theta_j}\,\mathrm{e}^{\delta(\theta_jx-\theta_ky)} & \text{if } x\in(-\infty,0],\\ 0 & \text{if } x\in[0,+\infty).\end{cases}$$

On the other hand we formally have that

$$\lim_{\mu\to+\infty}\varpi_\mu(t;x,y)=\lim_{\mu\to+\infty}\mathbb{E}_x(\mathrm{e}^{-\mu T(t)},X(t)\in\mathrm{d}y)/\mathrm{d}y=\mathbb{P}_x\Big\{\max_{0\le s\le t}X(s)<0,X(t)\in\mathrm{d}y\Big\}/\mathrm{d}y$$

from which we deduce that

$$\lim_{\mu\to+\infty}\varphi_{\lambda,\mu}(x,y)=\int_0^\infty\mathrm{e}^{-\lambda t}\Big[\mathbb{P}_x\Big\{\max_{0\le s\le t}X(s)<0,X(t)\in\mathrm{d}y\Big\}/\mathrm{d}y\Big]\mathrm{d}t.$$

Then, for $x,y\le 0$, we retrieve the pseudo-distribution of $(\max_{0\le s\le t}X(s),X(t))$ (through its Laplace transform with respect to t) displayed in [13].

As in the previous case, we need to expand the fractions lying in Proposition 3.3. Since the computations are quite similar to the previous case, we omit them and we only produce the second form for $\varphi_{\lambda,\mu}(x,y)$ below.

Proposition 3.4 *Suppose $y \le 0$. The function $\varphi_{\lambda,\mu}(x,y)$ can be written as follows:*

for $x \in (-\infty, 0]$,

$$\varphi_{\lambda,\mu}(x,y) = \varrho_\lambda(x,y) - \frac{1}{\lambda^{1-\frac{1}{N}}}\left[\sum_{j\in J,\,k\in K} \frac{\theta_j A_j \theta_k B_k}{\theta_k - \theta_j}\,\mathrm{e}^{\sqrt[N]{\lambda}\,(\theta_j x - \theta_k y)} - \sqrt[N]{\lambda} \sum_{j\in J,\,k,k'\in K} \frac{A_j \theta_k B_k B_{k'} D_{jkk'}}{\theta_j^{\#K-2}(\theta_k - \theta_j)}\,\frac{\mathrm{e}^{\sqrt[N]{\lambda}\,(\theta_j x - \theta_k y)}}{\theta_{k'}\sqrt[N]{\lambda+\mu} - \theta_j \sqrt[N]{\lambda}}\right];$$

for $x \in [0, +\infty)$,

$$\varphi_{\lambda,\mu}(x,y) = \frac{1\!\!1_{\{\#K=\#J+1\}}}{(\lambda+\mu)^{\frac{\#J}{N}}\lambda^{\frac{\#J}{N}}}\left[\sum_{k\in K}\theta_k B_k \mathrm{e}^{\theta_k\sqrt[N]{\lambda+\mu}\,x}\right]\left[\sum_{k\in K}\theta_k^{\#K} B_k \mathrm{e}^{-\theta_k\sqrt[N]{\lambda}\,y}\right] + \frac{1}{(\lambda+\mu)^{\frac{\#J-1}{N}}\lambda^{\frac{\#K-1}{N}}} \sum_{j\in J,\,k,k'\in K} \frac{\theta_j^{\#J-\#K+1} A_j B_k \theta_{k'} B_{k'} D_{jkk'}}{\theta_k^{\#J-1}(\theta_j - \theta_{k'})}\,\frac{\mathrm{e}^{\theta_k\sqrt[N]{\lambda+\mu}\,x - \theta_{k'}\sqrt[N]{\lambda}\,y}}{\theta_k\sqrt[N]{\lambda+\mu} - \theta_j\sqrt[N]{\lambda}}.$$

4 Inverting with respect to μ

In this section we invert the Laplace transform $\varphi_{\lambda,\mu}(x,y)$ with respect to μ. Let us recall that

$$\varphi_{\lambda,\mu}(x,y) = \int_0^\infty \mathrm{e}^{-\mu s}\psi_\lambda(s;x,y)\,\mathrm{d}s$$

with

$$\psi_\lambda(s;x,y) = \int_0^\infty \mathrm{e}^{-\lambda t}[\mathbb{P}_x\{T(t)\in\mathrm{d}s, X(t)\in\mathrm{d}y\}/(\mathrm{d}s\,\mathrm{d}y)]\,\mathrm{d}t.$$

As previously, we distinguish the two cases $y \ge 0$ and $y \le 0$.

4.1 Case $y \ge 0$

We need to invert the Laplace transform of the expressions of $\varphi_{\lambda,\mu}(x,y)$ lying in Proposition 3.2 with respect to μ. For this we shall make use of the identities

$$\frac{1}{\nu^\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^\infty \mathrm{e}^{-\nu s}s^{\alpha-1}\,\mathrm{d}s \quad \text{for } \alpha>0, \qquad \frac{1}{\nu^\alpha-\beta} = \int_0^\infty \mathrm{e}^{-\nu s}[s^{\alpha-1}E_{\alpha,\alpha}(\beta s^\alpha)]\,\mathrm{d}s \quad \text{for } \nu^\alpha > |\beta|$$

where $E_{a,b}(\xi) = \sum_{n=0}^\infty \frac{\xi^n}{\Gamma(an+b)}$ is the Mittag-Leffler function (see [7], p. 210). We also introduce, for any integer m such that $m \le N-1$, the function $I_m(s;\xi)$ characterized by its Laplace transform:

$$\int_0^\infty \mathrm{e}^{-\nu s} I_m(s;\xi)\,\mathrm{d}s = \frac{\mathrm{e}^{-\xi\sqrt[N]{\nu}}}{\nu^{m/N}} \quad \text{for } \mathrm{Re}(\xi)\ge 0.$$

That is, $I_m(s;\xi)$ is the inverse Laplace transform of the function $\nu \mapsto \nu^{-m/N}\mathrm{e}^{-\xi\sqrt[N]{\nu}}$. An expression for $I_m(s;\xi)$ can be found in [13] and is

$$I_m(s;\xi) = \frac{N\mathrm{i}}{2\pi}\left(\mathrm{e}^{-\mathrm{i}\frac{m}{N}\pi}\int_0^\infty \nu^{N-m-1}\mathrm{e}^{-s\nu^N - \mathrm{e}^{\mathrm{i}\frac{\pi}{N}}\xi\nu}\,\mathrm{d}\nu - \mathrm{e}^{\mathrm{i}\frac{m}{N}\pi}\int_0^\infty \nu^{N-m-1}\mathrm{e}^{-s\nu^N - \mathrm{e}^{-\mathrm{i}\frac{\pi}{N}}\xi\nu}\,\mathrm{d}\nu\right).$$

With this at hands, we extract

$$\frac{\mathrm{e}^{-\theta_j\sqrt[N]{\lambda+\mu}\,y}}{(\lambda+\mu)^{\frac{\#K}{N}}} = \int_0^\infty \mathrm{e}^{-\mu s}[\mathrm{e}^{-\lambda s}I_{\#K}(s;\theta_j y)]\,\mathrm{d}s$$

and

$$\frac{\mathrm{e}^{\sqrt[N]{\lambda+\mu}\,(\theta_k x-\theta_j y)}}{(\lambda+\mu)^{\frac{\alpha}{N}}(\sqrt[N]{\lambda+\mu}-\beta)} = \int_0^\infty \mathrm{e}^{-(\lambda+\mu)s} I_\alpha(s;\theta_j y-\theta_k x)\,\mathrm{d}s \times \int_0^\infty \mathrm{e}^{-(\lambda+\mu)s}\left[s^{\frac{1}{N}-1}E_{\frac{1}{N},\frac{1}{N}}\big(\beta\sqrt[N]{s}\big)\right]\mathrm{d}s$$
$$= \int_0^\infty \mathrm{e}^{-(\lambda+\mu)s}\left[\int_0^s \sigma^{\frac{1}{N}-1}E_{\frac{1}{N},\frac{1}{N}}\big(\beta\sqrt[N]{\sigma}\big)I_\alpha(s-\sigma;\theta_j y-\theta_k x)\,\mathrm{d}\sigma\right]\mathrm{d}s.$$

In particular, the latter formula provides for $\alpha = N-2$ and $\beta = \frac{\theta_{j'}}{\theta_k}\sqrt[N]{\lambda}$:

$$\frac{\mathrm{e}^{\sqrt[N]{\lambda+\mu}\,(\theta_k x-\theta_j y)}}{(\lambda+\mu)^{1-\frac{2}{N}}(\theta_k\sqrt[N]{\lambda+\mu}-\theta_{j'}\sqrt[N]{\lambda})}$$
$$= \frac{1}{\theta_k}\int_0^\infty \mathrm{e}^{-\mu s}\left[\mathrm{e}^{-\lambda s}\int_0^s \sigma^{\frac{1}{N}-1}E_{\frac{1}{N},\frac{1}{N}}\!\left(\frac{\theta_{j'}}{\theta_k}\sqrt[N]{\lambda\sigma}\right) I_{N-2}(s-\sigma;\theta_j y-\theta_k x)\,\mathrm{d}\sigma\right]\mathrm{d}s$$

and for $\alpha = \#J-1$ and $\beta = \frac{\theta_j}{\theta_k}\sqrt[N]{\lambda}$:

$$\frac{\mathrm{e}^{-\theta_{j'}\sqrt[N]{\lambda+\mu}\,y}}{(\lambda+\mu)^{\frac{\#J-1}{N}}(\theta_k\sqrt[N]{\lambda+\mu}-\theta_j\sqrt[N]{\lambda})}$$
$$= \frac{1}{\theta_k}\int_0^\infty \mathrm{e}^{-\mu s}\left[\mathrm{e}^{-\lambda s}\int_0^s \sigma^{\frac{1}{N}-1}E_{\frac{1}{N},\frac{1}{N}}\!\left(\frac{\theta_j}{\theta_k}\sqrt[N]{\lambda\sigma}\right) I_{\#J-1}(s-\sigma;\theta_{j'} y)\,\mathrm{d}\sigma\right]\mathrm{d}s.$$

Similar representations hold for $\frac{\mathrm{e}^{\sqrt[N]{\lambda+\mu}\,(\theta_k x-\theta_j y)}}{(\lambda+\mu)^{1-\frac{1}{N}}}$ and $\frac{\mathrm{e}^{\theta_k\sqrt[N]{\lambda+\mu}\,(x-y)}}{(\lambda+\mu)^{1-\frac{1}{N}}}$. Combining all these results, we obtain the proposition below.

Proposition 4.1 *Suppose $y \geq 0$. The function $\psi_\lambda(s;x,y)$ admits the following representation:*

for $x \in (-\infty, 0]$,

$$\psi_\lambda(s;x,y) = \kappa_N \mathrm{e}^{-\lambda s}\left[\frac{\mathbb{1}_{\{\#J=\#K+1\}}}{\lambda^{\frac{\#K}{N}}}\left(\sum_{j\in J}\theta_j A_j \mathrm{e}^{\theta_j\sqrt[N]{\lambda}\,x}\right)\left(\sum_{j\in J}\theta_j^{\#J}A_j I_{\#K}(s;\theta_j y)\right)\right.$$
$$+\frac{1}{\lambda^{\frac{\#K-1}{N}}}\sum_{j,j'\in J,\,k\in K}\frac{A_j\theta_{j'}A_{j'}\theta_k^{\#K-\#J}B_k C_{jj'k}}{\theta_j^{\#K-1}(\theta_{j'}-\theta_k)}\,\mathrm{e}^{\theta_j\sqrt[N]{\lambda}\,x}$$
$$\left.\times\int_0^s \sigma^{\frac{1}{N}-1}E_{\frac{1}{N},\frac{1}{N}}\!\left(\frac{\theta_j}{\theta_k}\sqrt[N]{\lambda\sigma}\right) I_{\#J-1}(s-\sigma;\theta_{j'} y)\,\mathrm{d}\sigma\right];$$

for $x \in [0,+\infty)$,

$$\psi_\lambda(s;x,y) = \mathrm{e}^{-\lambda s}\left[p(s;x-y)+\sum_{j\in J,\,k\in K}\frac{\theta_j A_j\theta_k B_k}{\theta_j-\theta_k}\,I_{N-1}(s;\theta_j y-\theta_k x)\right.$$
$$\left.+\kappa_N\sum_{j,j'\in J,\,k\in K}\frac{\theta_j A_j A_{j'} B_k C_{jj'k}}{\theta_k^{\#J-1}(\theta_j-\theta_k)}\int_0^s \sigma^{\frac{1}{N}-1}E_{\frac{1}{N},\frac{1}{N}}\!\left(\frac{\theta_{j'}}{\theta_k}\sqrt[N]{\lambda\sigma}\right) I_{N-2}(s-\sigma;\theta_j y-\theta_k x)\,\mathrm{d}\sigma\right].$$

Remark 4.1 *The foregoing representation of $\psi_\lambda(s;x,y)$ related to the case $x \geq 0$ involves the heat-kernel $p(s;x-y)$. We can write this latter by means of the function I_{N-1}. Indeed, using the definition of the function ϱ_λ, we see that for $\xi \geq 0$,*

$$\varrho_\lambda(\xi,0) = -\frac{1}{N\lambda^{1-\frac{1}{N}}}\sum_{k\in K}\theta_k \mathrm{e}^{\theta_k\sqrt[N]{\lambda}\,\xi} = -\int_0^\infty \mathrm{e}^{-\lambda t}\left(\frac{1}{N}\sum_{k\in K}\theta_k I_{N-1}(t;-\theta_k\xi)\right)\mathrm{d}t$$

which entails that

$$p(t;\xi) = -\frac{1}{N}\sum_{k\in K}\theta_k I_{N-1}(t;-\theta_k\xi). \tag{4.1}$$

Similarly, for $\xi \leq 0$,

$$p(t;\xi) = \frac{1}{N}\sum_{j\in J}\theta_j I_{N-1}(t;-\theta_j\xi).$$

4.2 Case $y \le 0$

Since the computations are quite similar, we only produce the result corresponding to the case where $y \le 0$.

Proposition 4.2 *Suppose $y \le 0$. The function $\psi_\lambda(s;x,y)$ admits the following representation:*

for $x \in (-\infty,0]$,

$$\psi_\lambda(s;x,y) = \left[\varrho_\lambda(x,y) + \frac{1}{\lambda^{1-\frac{1}{N}}} \sum_{j\in J,\, k\in K} \frac{\theta_j A_j \theta_k B_k}{\theta_j - \theta_k}\, \mathrm{e}^{\sqrt[N]{\lambda}\,(\theta_j x - \theta_k y)}\right]\delta_0(s)$$
$$- \frac{1}{\lambda^{1-\frac{2}{N}}} \sum_{j\in J,\, k,k'\in K} \frac{A_j \theta_k B_k B_{k'} D_{jkk'}}{\theta_j^{\#K-2}\theta_{k'}(\theta_k - \theta_j)}\, \mathrm{e}^{-\lambda s} s^{\frac{1}{N}-1} E_{\frac{1}{N},\frac{1}{N}}\!\left(\frac{\theta_j}{\theta_{k'}} \sqrt[N]{\lambda s}\right) \mathrm{e}^{\sqrt[N]{\lambda}\,(\theta_j x - \theta_k y)};$$

for $x \in [0,+\infty)$,

$$\psi_\lambda(s;x,y) = \frac{1\!\!1_{\{\#K=\#J+1\}}}{\lambda^{\frac{\#J}{N}}} \left[\sum_{k\in K} \theta_k B_k I_{\#J}(s;-\theta_k x)\right]\!\left[\sum_{k\in K} \theta_k^{\#K} B_k \mathrm{e}^{-\lambda s - \theta_k \sqrt[N]{\lambda}\, y}\right]$$
$$+ \frac{1}{\lambda^{\frac{\#K-1}{N}}} \sum_{j\in J,\, k,k'\in K} \frac{\theta_j^{\#J-\#K+1} A_j B_k \theta_{k'} B_{k'} D_{jkk'}}{\theta_k^{\#J}(\theta_j - \theta_{k'})}\, \mathrm{e}^{-\lambda s - \theta_{k'} \sqrt[N]{\lambda}\, y}$$
$$\times \int_0^s \sigma^{\frac{1}{N}-1} E_{\frac{1}{N},\frac{1}{N}}\!\left(\frac{\theta_j}{\theta_k} \sqrt[N]{\lambda\sigma}\right) I_{\#J-1}(s-\sigma;-\theta_k x)\,\mathrm{d}\sigma.$$

5 Inverting with respect to λ

This part is devoted to inverting the Laplace transform $\psi_\lambda(s;x,y)$ with respect to λ. We recall that

$$\psi_\lambda(s;x,y) = \int_0^\infty \mathrm{e}^{-\lambda t} \chi(s,t;x,y)\,\mathrm{d}t = \int_s^\infty \mathrm{e}^{-\lambda t} \chi(s,t;x,y)\,\mathrm{d}t$$

where $\chi(s,t;x,y) = \mathbb{P}_x\{T(t) \in \mathrm{d}s, X(t) \in \mathrm{d}y\}/(\mathrm{d}s\,\mathrm{d}y) \times 1\!\!1_{[0,t]}(s)$ is the quantity of main interest of this work. For this, we have to write $E_{\frac{1}{N},\frac{1}{N}}\!\left(\frac{\theta_j}{\theta_k}\sqrt[N]{\lambda\sigma}\right)$ as a Laplace transform with respect to λ. We note that, for any complex number β such that $|\beta| < \sqrt[N]{t}$,

$$\int_0^\infty \mathrm{e}^{-t\lambda}\, \lambda^{\frac{1}{N}-1} E_{\frac{1}{N},\frac{1}{N}}\big(\beta\sqrt[N]{\lambda}\big)\,\mathrm{d}\lambda = \sum_{p=0}^\infty \frac{\beta^p}{\Gamma\big(\frac{p+1}{N}\big)} \int_0^\infty \mathrm{e}^{-t\lambda}\, \lambda^{\frac{p+1}{N}-1}\,\mathrm{d}\lambda = \frac{1}{\sqrt[N]{t}} \sum_{p=0}^\infty \left(\frac{\beta}{\sqrt[N]{t}}\right)^p = \frac{1}{\sqrt[N]{t} - \beta}.$$

Then, by Bromwich's inversion formula, we have for any $c > |\beta|^N$

$$E_{\frac{1}{N},\frac{1}{N}}\big(\beta\sqrt[N]{\lambda}\big) = \frac{1}{2\mathrm{i}\pi} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\mathrm{e}^{\lambda t}}{t^{\frac{1}{N}} - \beta}\,\mathrm{d}t.$$

Suppose now that $\beta = \frac{\theta_j}{\theta_k}\sqrt[N]{\sigma}$ with $j \in J$ and $k \in K$. The possible singularities of the function $t \mapsto \frac{1}{t^{1/N}-\beta}$ satisfy $t = \beta^N = \sigma$. Thus the only possible singularity is σ. But $\sigma^{1/N} - \beta = \frac{\theta_k - \theta_j}{\theta_k}\sqrt[N]{\sigma} \neq 0$ which implies that the function $t \mapsto \frac{1}{t^{1/N}-\beta}$ has no singularity. So, we can shift the integration line $\mathrm{Re}\, t = c$ to the line $\mathrm{Re}\, t = 0$ and next refold this latter to a loop enclosing the half-line $\mathrm{Im}\, t = 0$, $\mathrm{Re}\, t \le 0$. Roughly speaking, this loop is defined as the union of the two half-lines $\mathrm{Im}\, t = 0^-$, $\mathrm{Re}\, t \le 0$ (from $-\infty$ to 0) and $\mathrm{Im}\, t = 0^+$, $\mathrm{Re}\, t \le 0$ (from 0 to $-\infty$). As a byproduct, we get

$$\begin{aligned}
E_{\frac{1}{N},\frac{1}{N}}\big(\beta\sqrt[N]{\lambda}\big) &= \frac{\lambda^{1-\frac{1}{N}}}{2\mathrm{i}\pi} \left[\int_{-\infty}^0 \frac{\mathrm{e}^{\lambda t}}{\mathrm{e}^{-\frac{\mathrm{i}\pi}{N}}\sqrt[N]{-t} - \beta}\,\mathrm{d}t - \int_{-\infty}^0 \frac{\mathrm{e}^{\lambda t}}{\mathrm{e}^{\frac{\mathrm{i}\pi}{N}}\sqrt[N]{-t} - \beta}\,\mathrm{d}t\right] \\
&= \frac{\lambda^{1-\frac{1}{N}}}{2\mathrm{i}\pi} \int_0^\infty \mathrm{e}^{-\lambda t} \left[\frac{1}{\mathrm{e}^{-\frac{\mathrm{i}\pi}{N}}\sqrt[N]{t} - \beta} - \frac{1}{\mathrm{e}^{\frac{\mathrm{i}\pi}{N}}\sqrt[N]{t} - \beta}\right]\mathrm{d}t \\
&= \frac{\sin(\frac{\pi}{N})\lambda^{1-\frac{1}{N}}}{\pi} \int_0^\infty \frac{\mathrm{e}^{-\lambda t}\sqrt[N]{t}}{t^{\frac{2}{N}} - 2\beta\cos(\frac{\pi}{N})t^{\frac{1}{N}} + \beta^2}\,\mathrm{d}t. \qquad (5.1)
\end{aligned}$$

We find it interesting to mention that when β is a negative number (obtained when $\theta_j = -\theta_k$), formula (5.1) is a particular case of a general representation of the Mittag-Leffler function which can be found in [3]. By integrating (5.1) by parts, we obtain

$$\begin{aligned}\int_0^\infty \frac{\mathrm{e}^{-\lambda t}\sqrt[N]{t}}{t^{\frac{2}{N}}-2\beta\cos(\frac{\pi}{N})t^{\frac{1}{N}}+\beta^2}\,\mathrm{d}t &= \int_0^\infty \frac{\sqrt[N]{t}}{t^{\frac{2}{N}}-2\beta\cos(\frac{\pi}{N})t^{\frac{1}{N}}+\beta^2}\,\mathrm{d}\Big(-\frac{\mathrm{e}^{-\lambda t}}{\lambda}\Big)\\ &= \frac{1}{N\lambda}\int_0^\infty \mathrm{e}^{-\lambda t}\,\frac{t^{\frac{1}{N}-1}(\beta^2-t^{\frac{2}{N}})}{(t^{\frac{2}{N}}-2\beta\cos(\frac{\pi}{N})t^{\frac{1}{N}}+\beta^2)^2}\,\mathrm{d}t.\end{aligned}$$

Finally, we derive the following representation for $\beta = \frac{\theta_j}{\theta_k}\sqrt[N]{\sigma}$:

$$E_{\frac{1}{N},\frac{1}{N}}\big(\beta\sqrt[N]{\lambda}\big) = \frac{1}{\sqrt[N]{\lambda}}\int_0^\infty \mathrm{e}^{-\lambda t} f(t;\beta)\,\mathrm{d}t$$

with

$$f(t;\beta) = \frac{\sin(\frac{\pi}{N})}{\pi N}\,\frac{t^{\frac{1}{N}-1}(\beta^2-t^{\frac{2}{N}})}{(t^{\frac{2}{N}}-2\beta\cos(\frac{\pi}{N})t^{\frac{1}{N}}+\beta^2)^2}.$$

5.1 Case $y \geq 0$

Assume that $y \geq 0$. Regarding the expressions of $\psi_\lambda(s;x,y)$ in Proposition 4.1, we see that we have to perform the following inversions with respect to λ:

$$\begin{aligned}&\mathrm{e}^{-\lambda s} = \int_0^\infty \mathrm{e}^{-\lambda t}\delta_s(t)\,\mathrm{d}t, \qquad \frac{\mathrm{e}^{-\lambda s+\theta_j\sqrt[N]{\lambda}\,x}}{\lambda^{\frac{\#K}{N}}} = \int_s^\infty \mathrm{e}^{-\lambda t} I_{\#K}(t-s;-\theta_j x)\,\mathrm{d}t,\\ &\frac{\mathrm{e}^{-\lambda s+\theta_j\sqrt[N]{\lambda}\,x}}{\lambda^{\frac{\#K-1}{N}}}\,E_{\frac{1}{N},\frac{1}{N}}\bigg(\frac{\theta_j}{\theta_k}\sqrt[N]{\lambda\sigma}\bigg) = \int_s^\infty \mathrm{e}^{-\lambda t}\left[\int_0^{t-s} I_{\#K}(\tau;-\theta_j x)\,f\Big(t-s-\tau;\frac{\theta_j}{\theta_k}\sqrt[N]{\sigma}\Big)\,\mathrm{d}\tau\right]\mathrm{d}t,\\ &\mathrm{e}^{-\lambda s}E_{\frac{1}{N},\frac{1}{N}}\bigg(\frac{\theta_{j'}}{\theta_k}\sqrt[N]{\lambda\sigma}\bigg) = \int_s^\infty \mathrm{e}^{-\lambda t}\left[\frac{1}{\Gamma(\frac{1}{N})}\int_0^{t-s}(t-s-\tau)^{\frac{1}{N}-1} f\Big(\tau,\frac{\theta_{j'}}{\theta_k}\sqrt[N]{\sigma}\Big)\,\mathrm{d}\tau\right]\mathrm{d}t.\end{aligned}$$

By using all these identities, we can extract the ppdf $\chi(s,t;x,y)$.

Theorem 5.1 *The pseudo-distribution of $(T(t),X(t))$ is given by the following formulas: for $s \in [0,t]$ and $y \geq 0$,*

if $x \in (-\infty,0]$,

$$\begin{aligned}\chi(s,t;x,y) = {}&\kappa_N \mathbb{1}_{\{\#J=\#K+1\}}\left[\sum_{j\in J}\theta_j A_j I_{\#K}(t-s;-\theta_j x)\right]\left[\sum_{j\in J}\theta_j^{\#J}A_j I_{\#K}(s;\theta_j y)\right]\\ &+\kappa_N\sum_{j,j'\in J,\,k\in K}\frac{A_j\theta_{j'}A_{j'}\theta_k^{\#K-\#J}B_k C_{jj'k}}{\theta_j^{\#K-1}(\theta_{j'}-\theta_k)}\int_0^s \sigma^{\frac{1}{N}-1}I_{\#J-1}(s-\sigma;\theta_{j'}y)\,\mathrm{d}\sigma\\ &\times\int_0^{t-s} I_{\#K}(\tau;-\theta_j x)\,f\Big(t-s-\tau;\frac{\theta_j}{\theta_k}\sqrt[N]{\sigma}\Big)\,\mathrm{d}\tau;\end{aligned}$$

if $x \in [0,+\infty)$,

$$\begin{aligned}\chi(s,t;x,y) = {}&\left[p(t;x-y)+\sum_{j\in J,\,k\in K}\frac{\theta_j A_j\theta_k B_k}{\theta_j-\theta_k}\,I_{N-1}(t;\theta_j y-\theta_k x)\right]\delta_s(t)\\ &+\kappa_N\sum_{j,j'\in J,\,k\in K}\frac{\theta_j A_j A_{j'}B_k C_{jj'k}}{\theta_k^{\#J-1}(\theta_j-\theta_k)}\int_0^s \sigma^{\frac{1}{N}-1}I_{N-2}(s-\sigma;\theta_j y-\theta_k x)\,\mathrm{d}\sigma\\ &\times\int_0^{t-s}\frac{(t-s-\tau)^{\frac{1}{N}-1}}{\Gamma(\frac{1}{N})}f\Big(\tau;\frac{\theta_{j'}}{\theta_k}\sqrt[N]{\sigma}\Big)\,\mathrm{d}\tau.\end{aligned}$$

Remark 5.1 *For $x = 0$, Theorem 5.1 yields a representation of the pseudo-distribution of $(T(t),X(t))$ under $\mathbb{P}_0$ which apparently differs from that we obtained in [5] (formula (6.1)) by using Spitzer's identity. It seems difficult to prove directly the equality of both representations without comparing their Laplace transforms. We made this last comparison in Remark 3.3.*

Remark 5.2 *The term multiplied by the atom $\delta_s(t)$ in the last case (i.e. when $x \geq 0$) of Theorem 5.1 yields for $x, y \geq 0$:*

$$\mathbb{P}_x\{T(t)=t, X(t)\in \mathrm{d}y\}/\mathrm{d}y = \mathbb{P}_x\Big\{\min_{0\leq s\leq t} X(s)\geq 0, X(t)\in \mathrm{d}y\Big\}/\mathrm{d}y$$
$$= p(t;x-y) + \sum_{j\in J,\, k\in K} \frac{\theta_j A_j \theta_k B_k}{\theta_j-\theta_k}\, I_{N-1}(t;\theta_j y-\theta_k x). \tag{5.2}$$

The quantity above is nothing but the distribution of the pseudo-process $(X(t))_{t\geq 0}$ killed when overshooting level 0 from below, that is $\mathbb{P}_x\{X(t)\in \mathrm{d}y, \tau_0^-\geq t\}/\mathrm{d}y$ where $\tau_0^- = \inf\{t\geq 0 : X(t)<0\}$. Moreover, by integrating (5.2) with respect to y on $(0,+\infty)$, we obtain

$$\mathbb{P}_x\{T(t)=t\} = \int_{-\infty}^{x} p(t;\xi)\,\mathrm{d}\xi + \sum_{j\in J,\, k\in K} \frac{A_j\theta_k B_k}{\theta_j-\theta_k}\int_{-\theta_k x}^{+\infty} I_{N-1}(t;\zeta)\,\mathrm{d}\zeta. \tag{5.3}$$

In the last integral above, the integration path is a half-line in the complex plane going from $-\theta_k x$ to infinity in the direction of the positive real axis. Using the definition of the function I_p, we see that

$$\int_0^{\infty} \mathrm{e}^{-\lambda t}\bigg(\int_{\xi}^{+\infty} I_p(t;\zeta)\,\mathrm{d}\zeta\bigg)\mathrm{d}t = \frac{1}{\lambda^{\frac{p}{N}}}\int_{\xi}^{+\infty}\mathrm{e}^{-\sqrt[N]{\lambda}\,\zeta}\,\mathrm{d}\zeta = \frac{\mathrm{e}^{-\sqrt[N]{\lambda}\,\xi}}{\lambda^{\frac{p+1}{N}}} = \int_0^{\infty}\mathrm{e}^{-\lambda t} I_{p+1}(t;\xi)\,\mathrm{d}t$$

from which we extract

$$\int_{\xi}^{+\infty} I_p(t;\zeta)\,\mathrm{d}\zeta = I_{p+1}(t;\xi). \tag{5.4}$$

Then, by (4.1) and (5.4), for $x\leq 0$,

$$\int_{-\infty}^{x} p(t;\xi)\,\mathrm{d}\xi = 1-\int_{x}^{+\infty} p(t;\xi)\,\mathrm{d}\xi = 1-\frac{1}{N}\sum_{k\in K}\int_{-\theta_k x}^{+\infty} I_{N-1}(t;\zeta)\,\mathrm{d}\zeta = 1-\frac{1}{N}\sum_{k\in K} I_N(t;-\theta_k x). \tag{5.5}$$

As a result, using (5.3), (5.4) and (5.5),

$$\mathbb{P}_x\{T(t)=t\} = 1-\frac{1}{N}\sum_{k\in K} I_N(t;-\theta_k x) + \sum_{j\in J,\, k\in K}\frac{A_j\theta_k B_k}{\theta_j-\theta_k}\, I_N(t;-\theta_k x)$$
$$= 1+\sum_{k\in K}\Bigg(\theta_k B_k\sum_{j\in J}\frac{A_j}{\theta_j-\theta_k}-\frac{1}{N}\Bigg) I_N(t;-\theta_k x). \tag{5.6}$$

Finally, thanks to (2.10) of [13], we have

$$\theta_k\sum_{j\in J}\frac{A_j}{\theta_j-\theta_k} = \sum_{j\in J}\bigg(\frac{\theta_j A_j}{\theta_j-\theta_k}-A_j\bigg) = \frac{1}{NB_k}-1$$

and, plugging this into (5.6), we get the pseudo-distribution of τ_0^-: for $x\geq 0$,

$$\mathbb{P}_x\{\tau_0^-\leq t\} = \sum_{k\in K} B_k I_N(t;-\theta_k x).$$

In order to retrieve the ppdf of τ_0^- (see [13]), we need to differentiate the function I_N with respect to time. We have

$$\int_0^{\infty}\mathrm{e}^{-\lambda t}\frac{\partial I_p}{\partial t}(t;\xi)\,\mathrm{d}t = \lambda\int_0^{\infty}\mathrm{e}^{-\lambda t} I_p(t;\xi)\,\mathrm{d}t - I_p(0;\xi) = \frac{\mathrm{e}^{-\sqrt[N]{\lambda}\,\xi}}{\lambda^{\frac{p}{N}-1}} = \int_0^{\infty}\mathrm{e}^{-\lambda t} I_{p-N}(t;\xi)\,\mathrm{d}t$$

and then

$$\frac{\partial I_p}{\partial t}(t;\xi) = I_{p-N}(t;\xi).$$

The ppdf of τ_0^- is thus given, for $x\geq 0$, by

$$\mathbb{P}_x\{\tau_0^-\in \mathrm{d}t\}/\mathrm{d}t = \sum_{k\in K} B_k I_0(t;-\theta_k x).$$

5.2 Case $y \le 0$

We now assume that $y \le 0$. Regarding the expression of $\psi_\lambda(s;x,y)$ in Proposition 4.2, we see that we need the following inversions with respect to λ:

$$\frac{\mathrm{e}^{\sqrt[N]{\lambda}\,(\theta_j x-\theta_k y)}}{\lambda^{1-\frac{1}{N}}}=\int_0^\infty \mathrm{e}^{-\lambda t} I_{N-1}(t;\theta_k y-\theta_j x)\,\mathrm{d}t,\quad \frac{\mathrm{e}^{-\lambda s-\theta_k\sqrt[N]{\lambda}\,y}}{\lambda^{\frac{\#J}{N}}}=\int_s^\infty \mathrm{e}^{-\lambda t} I_{\#J}(t-s;\theta_k y)\,\mathrm{d}t,$$

$$\frac{\mathrm{e}^{-\lambda s-\theta_{k'}\sqrt[N]{\lambda}\,y}}{\lambda^{\frac{\#K-1}{N}}}\,E_{\frac{1}{N},\frac{1}{N}}\!\left(\frac{\theta_j}{\theta_k}\sqrt[N]{\lambda\sigma}\right)=\int_s^\infty \mathrm{e}^{-\lambda t}\left[\int_0^{t-s} I_{\#K}(\tau;\theta_{k'}y)\,f\Big(t-s-\tau;\frac{\theta_j}{\theta_k}\sqrt[N]{\sigma}\Big)\,\mathrm{d}\tau\right]\mathrm{d}t,$$

$$\frac{\mathrm{e}^{-\lambda s+\sqrt[N]{\lambda}\,(\theta_j x-\theta_k y)}}{\lambda^{1-\frac{2}{N}}}\,E_{\frac{1}{N},\frac{1}{N}}\!\left(\frac{\theta_j}{\theta_{k'}}\sqrt[N]{\lambda s}\right)=\int_s^\infty \mathrm{e}^{-\lambda t}\left[\int_0^{t-s} I_{N-1}(\tau;\theta_k y-\theta_j x)\,f\Big(t-s-\tau;\frac{\theta_j}{\theta_{k'}}\sqrt[N]{s}\Big)\,\mathrm{d}\tau\right]\mathrm{d}t.$$

As a result, we derive the ppdf $\chi(s,t;x,y)$.

Theorem 5.2 *The pseudo-distribution of $(T(t),X(t))$ is given by the following formulas; for $s\in[0,t]$ and $y\le 0$,*

if $x\in(-\infty,0]$,

$$\chi(s,t;x,y)=\left[p(t;x-y)+\sum_{j\in J,\,k\in K}\frac{\theta_j A_j\theta_k B_k}{\theta_j-\theta_k}\,I_{N-1}(t;\theta_k y-\theta_j x)\right]\delta_0(s)$$
$$-\sum_{j\in J,\,k,k'\in K}\frac{A_j\theta_k B_k B_{k'}D_{jkk'}}{\theta_j^{\#K-2}\theta_{k'}(\theta_k-\theta_j)}\,s^{\frac{1}{N}-1}\int_0^{t-s} I_{N-1}(\tau;\theta_k y-\theta_j x)\,f\Big(t-s-\tau;\frac{\theta_j}{\theta_{k'}}\sqrt[N]{s}\Big)\,\mathrm{d}\tau,$$

if $x\in[0,+\infty)$,

$$\chi(s,t;x,y)=\mathbb{1}_{\{\#K=\#J+1\}}\left[\sum_{k\in K}\theta_k B_k I_{\#J}(s;-\theta_k x)\right]\left[\sum_{k\in K}\theta_k^{\#K}B_k I_{\#J}(t-s;\theta_k y)\right]$$
$$+\sum_{j\in J,\,k,k'\in K}\frac{\theta_j^{\#J-\#K+1}A_j B_k\theta_{k'}B_{k'}D_{jkk'}}{\theta_k^{\#J}(\theta_j-\theta_{k'})}\int_0^s \sigma^{\frac{1}{N}-1}I_{\#J-1}(s-\sigma;-\theta_k x)\,\mathrm{d}\sigma$$
$$\times\int_0^{t-s} I_{\#K}(\tau;\theta_{k'}y)f\Big(t-s-\tau;\frac{\theta_j}{\theta_k}\sqrt[N]{\sigma}\Big)\,\mathrm{d}\tau.$$

Remark 5.3 *The term multiplied by the atom $\delta_0(s)$ in the first case (i.e. when $x\le 0$) of Theorem 5.2 yields for $x,y\le 0$:*

$$\mathbb{P}_x\{T(t)=0,X(t)\in\mathrm{d}y\}/\mathrm{d}y=\mathbb{P}_x\Big\{\max_{0\le s\le t}X(s)\le 0,X(t)\in\mathrm{d}y\Big\}/\mathrm{d}y$$
$$=p(t;x-y)+\sum_{j\in J,\,k\in K}\frac{\theta_j A_j\theta_k B_k}{\theta_j-\theta_k}\,I_{N-1}(t;\theta_k y-\theta_j x).$$

This is the distribution of the pseudo-process $(X(t))_{t\ge 0}$ killed when overshooting level 0 from above, that is $\mathbb{P}_x\{X(t)\in\mathrm{d}y,\tau_0^+\ge t\}/\mathrm{d}y$ where $\tau_0^+=\inf\{t\ge 0:X(t)>0\}$. As in Remark 5.2, it may be seen that

$$\mathbb{P}_x\{\tau_0^+\in\mathrm{d}t\}/\mathrm{d}t=\sum_{j\in J}A_j I_0(t;-\theta_j x).$$

Remark 5.4 *It is possible to check the following identity: for $x,y\ge 0$,*

$$\sum_{j,j'\in J,\,k\in K}\frac{\theta_j A_j A_{j'}B_k C_{jj'k}}{\theta_k^{\#J-1}(\theta_j-\theta_k)}\int_0^s \sigma^{\frac{1}{N}-1}I_{N-2}(s-\sigma;\theta_j y-\theta_k x)\,\mathrm{d}\sigma\int_0^{t-s}\frac{(t-s-\tau)^{\frac{1}{N}-1}}{\Gamma(\frac{1}{N})}f\Big(\tau;\frac{\theta_{j'}}{\theta_k}\sqrt[N]{\sigma}\Big)\,\mathrm{d}\tau$$
$$=-\sum_{j,j'\in J,\,k\in K}\frac{\theta_j A_j A_{j'}B_k C_{jj'k}}{\theta_{j'}\theta_k^{\#J-2}(\theta_j-\theta_k)}\,(t-s)^{\frac{1}{N}-1}\int_0^s I_{N-1}(s-\sigma;\theta_j y-\theta_k x)\,f\Big(\sigma;\frac{\theta_k}{\theta_{j'}}\sqrt[N]{t-s}\Big)\,\mathrm{d}\sigma.\qquad(5.7)$$

This means that the sum involving the double integral (with respect to σ and τ) lying in the two last cases of Theorem 5.1 can be reduced into a sum involving only a single integral. The foregoing equality can *be proved directly by using Laplace transform and some algebra; see Appendix C for the details. Using similar algebra, we could check the following equality: if $x \le 0 \le y$,*

$$\sum_{j,j'\in J,\, k\in K} \frac{A_j\theta_{j'}A_{j'}\theta_k^{\#K-\#J}B_kC_{jj'k}}{\theta_j^{\#K-1}(\theta_{j'}-\theta_k)} \times \int_0^s \sigma^{\frac{1}{N}-1} I_{\#J-1}(s-\sigma;\theta_{j'}y)\,\mathrm{d}\sigma \int_0^{t-s} I_{\#K}(\tau;-\theta_j x)\, f\Big(t-s-\tau;\frac{\theta_j}{\theta_k}\sqrt[N]{\sigma}\Big)\,\mathrm{d}\tau$$

$$= -\sum_{j,j'\in J,\, k\in K} \frac{A_j\theta_{j'}A_{j'}\theta_k^{\#K-\#J+1}B_kC_{jj'k}}{\theta_j^{\#K}(\theta_{j'}-\theta_k)} \times \int_0^{t-s} \sigma^{\frac{1}{N}-1} I_{\#K-1}(t-s-\sigma;-\theta_j x)\,\mathrm{d}\sigma \int_0^{s} I_{\#J}(\tau;\theta_{j'}y) f\Big(s-\tau;\frac{\theta_k}{\theta_j}\sqrt[N]{\sigma}\Big)\,\mathrm{d}\tau.$$

Nonetheless, in this last case, both sums involve double integrals without any simplification. An explanation for these identities can be found in duality. In [11] the dual pseudo-process $X^* = -X$ of X is introduced. In the case where N is even, the pseudo-processes X and X^* have the same pseudo-distributions, while, in the case where N is odd, if we denote by X^+ (resp. X^-) the pseudo-process associated with $\kappa_N = +1$ (resp. -1), we have the identities in distribution $(X^+)^* = X^-$ and $(X^-)^* = X^+$. Let us introduce analogous notations for the settings $J, K, \theta_j, \theta_k, A_j, B_k, C_{jj'k}, D_{jkk'}$: when N is even, the settings $J, K, \theta_j, \theta_k, A_j, B_k, C_{jj'k}, D_{jkk'}$ are interchanged into $K, J, -\theta_k, -\theta_j, B_k, A_j, D_{jkk'}, C_{jj'k}$, while, when N is odd, the settings $J^+, K^+, \theta_j^+, \theta_k^+, A_j^+, B_k^+, C_{jj'k}^+, D_{jkk'}^+$ are interchanged into $K^-, J^-, -\theta_k^-, -\theta_j^-, B_k^-, A_j^-, D_{jkk'}^-, C_{jj'k}^-$ (where the superscripts refer to $\kappa_N = \pm 1$). As in [11], we have $\chi(s,t;x,y) = \chi(t-s,t;-x,-y)$ when N is even and $\chi^{\pm}(s,t;x,y) = \chi^{\mp}(t-s,t;-x,-y)$ when N is odd; this explains formula (5.7).

Remark 5.5 *By integrating each formula of Theorem 5.1 and Theorem 5.2 with respect to y, we can write out a representation for the marginal ppdf of $T(t)$. In Remarks 5.2 and 5.3 we have already displayed the parts associated with the atoms 0 and t. For $s \in (0,t)$, the continuous part is given by*

if $x \in (-\infty, 0]$,

$$\begin{aligned}\mathbb{P}_x\{T(t)\in \mathrm{d}s\}/\mathrm{d}s = {} & \sum_{j\in J,\, k,k'\in K} \frac{A_jB_kB_{k'}D_{jkk'}}{\theta_j^{\#K-2}\theta_{k'}(\theta_k-\theta_j)}\, s^{\frac{1}{N}-1} \int_0^{t-s} I_N(\tau;-\theta_j x)\, f\Big(t-s-\tau;\frac{\theta_j}{\theta_{k'}}\sqrt[N]{s}\Big)\,\mathrm{d}\tau \\ & + \frac{\kappa_N}{\Gamma(\frac{\#J}{N})} \sum_{j,j'\in J,\, k\in K} \frac{A_jA_{j'}\theta_k^{\#K-\#J}B_kC_{jj'k}}{\theta_j^{\#K-1}(\theta_{j'}-\theta_k)} \int_0^s \frac{\sigma^{\frac{1}{N}-1}}{(s-\sigma)^{\frac{\#K}{N}}}\,\mathrm{d}\sigma \\ & \times \int_0^{t-s} I_{\#K}(\tau;-\theta_j x)\, f\Big(t-s-\tau;\frac{\theta_j}{\theta_k}\sqrt[N]{\sigma}\Big)\,\mathrm{d}\tau,\end{aligned}$$

if $x \in [0, +\infty)$,

$$\begin{aligned}\mathbb{P}_x\{T(t)\in \mathrm{d}s\}/\mathrm{d}s = {} & -\kappa_N \sum_{j,j'\in J,\, k\in K} \frac{A_jA_{j'}B_kC_{jj'k}}{\theta_{j'}\theta_k^{\#J-2}(\theta_j-\theta_k)}\, (t-s)^{\frac{1}{N}-1} \int_0^{s} I_N(\sigma;-\theta_k x)\, f\Big(s-\sigma;\frac{\theta_k}{\theta_{j'}}\sqrt[N]{t-s}\Big)\,\mathrm{d}\sigma \\ & - \frac{1}{\Gamma(\frac{\#K+1}{N})} \sum_{j\in J,\, k,k'\in K} \frac{\theta_j^{\#J-\#K+1}A_jB_kB_{k'}D_{jkk'}}{\theta_k^{\#J}\theta_{k'}(\theta_j-\theta_{k'})} \int_0^s \sigma^{\frac{1}{N}-1} I_{\#J-1}(s-\sigma;-\theta_k x)\,\mathrm{d}\sigma \\ & \times \int_0^{t-s} \tau^{\frac{1-\#J}{N}} f\Big(t-s-\tau;\frac{\theta_j}{\theta_k}\sqrt[N]{\sigma}\Big)\,\mathrm{d}\tau.\end{aligned}$$

In the computations, we have made use of (5.4), (5.7), $I_p(t;0) = t^{\frac{p}{N}-1}/\Gamma(\frac{p}{N})$ as well as of the following equalities:

$$\int_0^{+\infty} \Big(\sum_{j\in J} \theta_j^{\#J} A_j I_{\#K}(s;\theta_j y)\Big)\mathrm{d}y = \Big(\sum_{j\in J}\theta_j^{\#J-1}A_j\Big) I_{\#K+1}(s;0) = 0,$$

$$\int_{-\infty}^{0} \Big(\sum_{k\in K} \theta_k^{\#K} B_k I_{\#J}(t-s;\theta_k y)\Big)\mathrm{d}y = -\Big(\sum_{k\in K}\theta_k^{\#K-1}B_k\Big) I_{\#J+1}(t-s;0) = 0.$$

For $x = 0$, we obtain an a priori intricate expression of the pseudo-probability $\mathbb{P}_0\{T(t)\in \mathrm{d}s\}/\mathrm{d}s$. Actually, it is known that this latter admits a very simple representation (a Beta law, see, e.g., [11]). Nevertheless, it seems difficult to check directly (without computing any Laplace transform) the equality of both representations.

Remark 5.6 *Let us introduce the last time above level* 0 *before time* t *as well as the first time above level* 0 *after time* t*:*

$$\sigma(t) = \sup\{s \in [0,t] : X(s) \geq 0\}, \quad \varsigma(t) = \inf\{s \geq t : X(s) \geq 0\}.$$

We are dealing with an excursion under level 0 *straddling time* t *for the pseudo-process* $(X(s))_{s\geq 0}$. *Using the pseudo-Markov property, we easily get the following relationship between the couple* $(\sigma(t), \varsigma(t))$ *and the family* $(T(s))_{s\geq 0}$*: for* $\sigma \leq t \leq \varsigma$,

$$\begin{aligned}\mathbb{P}_x\{\sigma(t) \leq \sigma, \varsigma(t) \geq \varsigma\} &= \mathbb{P}_x\Big\{\sup_{s\in[\sigma,\varsigma]} X(s) \leq 0\Big\} = \mathbb{E}_x\big(\mathbb{P}_{X(\sigma)}\{T(\varsigma-\sigma)=0\}1\!\!1_{\{X(\sigma)\leq 0\}}\big)\\ &= \int_{-\infty}^0 p(\sigma; x-\xi)\,\mathbb{P}_\xi\{T(\varsigma-\sigma)=0\}\,\mathrm{d}\xi\end{aligned}$$

with

$$\mathbb{P}_\xi\{T(\varsigma-\sigma)=0\} = 1 - \sum_{j\in J} A_j I_0(\varsigma-\sigma; -\theta_j\xi).$$

Appendix

A Proof of (3.7), (3.8) and (3.9)

Suppose that $y = 0$. By Proposition 3.1, we have for $x \leq 0$

$$\varphi_{\lambda,\mu}(x,0) = \frac{\kappa_N}{(\gamma\delta)^{\#J-1}} \sum_{j\in J} \theta_j A_j \left[\sum_{j'\in J} \theta_{j'} A_{j'} \prod_{j''\in J\setminus\{j\}} (\theta_{j'}\gamma - \theta_{j''}\delta)\right] \frac{\mathrm{e}^{\theta_j\delta x}}{\prod_{k\in K}(\theta_j\delta - \theta_k\gamma)}.$$

Let us expand the product below:

$$\prod_{j''\in J\setminus\{j\}} (\theta_{j'}\gamma - \theta_{j''}\delta) = (\theta_{j'}\gamma)^{\#J-1} + \sum_{p=0}^{\#J-2} c_p(\theta_{j'}\gamma)^p\delta^{\#J-1-p}$$

where the c_p, $0 \leq p \leq \#J-2$ are some coefficients depending on the $\theta_{j''}$, $j'' \in J\setminus\{j\}$, but not on the index j'. Recalling that the $A_{j'}$, $j' \in J$, solve a Vandermonde system, we have $\sum_{j'\in J} \theta_{j'}^p A_{j'} = 0$ for $1 \leq p \leq \#J-1$, $\sum_{j'\in J} \theta_{j'}^{\#J} A_{j'} = \kappa_N$ and then

$$\sum_{j'\in J} \theta_{j'} A_{j'} \prod_{j''\in J\setminus\{j\}} (\theta_{j'}\gamma - \theta_{j''}\delta) = \left(\sum_{j'\in J} \theta_{j'}^{\#J} A_{j'}\right)\gamma^{\#J-1} + \sum_{p=0}^{\#J-2} c_p \left(\sum_{j'\in J} \theta_{j'}^{p+1} A_{j'}\right)\gamma^p\delta^{\#J-1-p} = \kappa_N\gamma^{\#J-1}$$

On the other hand, by (2.1) we have $\prod_{i\in I}(\theta_j\delta - \theta_i\gamma) = (\theta_j\delta)^N - \kappa_N\gamma^N = -\kappa_N(\gamma^N - \delta^N)$, and then

$$\prod_{k\in K}(\theta_j\delta - \theta_k\gamma) = \frac{\prod_{i\in I}(\theta_j\delta - \theta_i\gamma)}{\prod_{j'\in J}(\theta_j\delta - \theta_{j'}\gamma)} = \frac{\mu}{\delta^{\#J}\prod_{j'\in J}(\theta_{j'}\frac{\gamma}{\delta} - \theta_j)}.$$

As a byproduct, for $x \leq 0$,

$$\varphi_{\lambda,\mu}(x,0) = \frac{\sqrt[N]{\lambda}}{\mu} \sum_{j\in J} \theta_j A_j \left[\prod_{j'\in J}\left(\theta_{j'}\sqrt[N]{\frac{\lambda+\mu}{\lambda}} - \theta_j\right)\right] \mathrm{e}^{\theta_j\sqrt[N]{\lambda}\,x}$$

which proves (3.7). Choosing now $x = 0$ yields

$$\varphi_{\lambda,\mu}(0,0) = \frac{\delta}{\mu} \sum_{j\in J} \theta_j A_j \left[\prod_{j'\in J}\left(\theta_{j'}\frac{\gamma}{\delta} - \theta_j\right)\right].$$

The product lying in the last displayed equation can be expanded into

$$\prod_{j'\in J}\left(\theta_{j'}\frac{\gamma}{\delta} - \theta_j\right) = (-1)^{\#J}\left[\theta_j^{\#J} - \left(\sum_{j'\in J}\theta_{j'}\right)\theta_j^{\#J-1}\frac{\gamma}{\delta} + \sum_{p=0}^{\#J-2} c'_p\theta_j^p\left(\frac{\gamma}{\delta}\right)^{\#J-p}\right]$$

where the c'_p, $0 \le p \le \#J-2$ are some coefficients depending on the $\theta_{j'}$, $j' \in J$, but not on the index j. As previously, with the additional aid of $\sum_{j\in J} \theta_j^{\#J+1} A_j = \kappa_{_N} \sum_{j\in J} \theta_j$, we get

$$\sum_{j\in J} \theta_j A_j \left[\prod_{j'\in J} \left(\theta_{j'} \frac{\gamma}{\delta} - \theta_j\right)\right] = (-1)^{\#J} \left[\sum_{j\in J} \theta_j^{\#J+1} A_j - \left(\sum_{j'\in J} \theta_{j'}\right)\left(\sum_{j\in J} \theta_j^{\#J} A_j\right)\frac{\gamma}{\delta}\right] = \left(\sum_{j\in J} \theta_j\right)\frac{\gamma-\delta}{\delta}$$

and then

$$\varphi_{\lambda,\mu}(0,0) = \left(\sum_{j\in J} \theta_j\right) \frac{\sqrt[N]{\lambda+\mu} - \sqrt[N]{\lambda}}{\mu}.$$

This proves (3.8).

On the other hand, we have for $x \ge 0$

$$\varphi_{\lambda,\mu}(x,0) = \frac{1}{\gamma^{N-1}} \sum_{k\in K} \theta_k B_k \left[\sum_{j\in J} \frac{\theta_j A_j}{\theta_j - \theta_k} \prod_{j'\in J} \left(\frac{\theta_j \gamma - \theta_{j'}\delta}{\theta_k \gamma - \theta_{j'}\delta}\right) - \frac{1}{NB_k}\right] \mathrm{e}^{\theta_k \gamma x}.$$

The sum lying within the brackets in the above equality can be written as follows:

$$\sum_{j\in J} \frac{\theta_j A_j}{\theta_j - \theta_k} \prod_{j'\in J} \left(\frac{\theta_j \gamma - \theta_{j'}\delta}{\theta_k \gamma - \theta_{j'}\delta}\right) = \frac{1}{\prod_{j'\in J}(\theta_k\gamma - \theta_{j'}\delta)} \sum_{j\in J} \frac{\theta_j A_j}{\theta_j - \theta_k} \prod_{j'\in J} (\theta_j\gamma - \theta_{j'}\delta).$$

First, we observe that

$$\prod_{j'\in J} (\theta_k\gamma - \theta_{j'}\delta) = \frac{\prod_{i\in I}(\theta_k\gamma - \theta_i\delta)}{\prod_{k'\in K}(\theta_k\gamma - \theta_{k'}\delta)} = \kappa_{_N} \frac{\mu}{\prod_{k'\in K}(\theta_k\gamma - \theta_{k'}\delta)}.$$

Second, invoking Formula (2.9) of [13]:

$$\sum_{j\in J} \frac{\theta_j A_j P(\theta_j)}{\theta_j - X} = \frac{P(X)}{\prod_{j\in J}(\theta_j - X)} + \kappa_{_N} c$$

which is valid for any polynomial P of degree $\#J$ and coefficient of highest degree c, we obtain for $P(X) = \prod_{j'\in J}(\gamma X - \theta_{j'}\delta)$ and $X = \theta_k$:

$$\sum_{j\in J} \frac{\theta_j A_j}{\theta_j - \theta_k} \prod_{j'\in J} (\theta_j\gamma - \theta_{j'}\delta) = \frac{\prod_{j'\in J}(\theta_k\gamma - \theta_{j'}\delta)}{\prod_{j\in J}(\theta_j - \theta_k)} + \kappa_{_N} \gamma^{\#J}$$

which implies, in view of (2.1), and since $\prod_{j\in J}(\theta_j - \theta_k) = NB_k$,

$$\sum_{j\in J} \frac{\theta_j A_j}{\theta_j - \theta_k} \prod_{j'\in J} \left(\frac{\theta_j \gamma - \theta_{j'}\delta}{\theta_k \gamma - \theta_{j'}\delta}\right) = \frac{1}{NB_k} + \frac{\gamma^{\#J}}{\mu} \prod_{k'\in K} (\theta_k\gamma - \theta_{k'}\delta)$$

As a result, for $x \ge 0$,

$$\varphi_{\lambda,\mu}(x,0) = \frac{\sqrt[N]{\lambda+\mu}}{\mu} \sum_{k\in K} \theta_k B_k \left[\prod_{k'\in K} \left(\theta_k - \theta_{k'} \sqrt[N]{\frac{\lambda}{\lambda+\mu}}\right)\right] \mathrm{e}^{\theta_k \sqrt[N]{\lambda+\mu}\, x}.$$

This is exactly (3.9). ■

B Proof of (3.10)

Suppose that $x = 0$ and $y \ge 0$. Proposition 3.1 yields

$$\varphi_{\lambda,\mu}(0,y) = \frac{1}{\gamma^{N-1}} \sum_{j\in J} \theta_j A_j \left[\sum_{k\in K} \frac{\theta_k B_k}{\theta_j - \theta_k} \prod_{j'\in J} \left(\frac{\theta_j \gamma - \theta_{j'}\delta}{\theta_k \gamma - \theta_{j'}\delta}\right) + \frac{1}{NA_j}\right] \mathrm{e}^{-\theta_j \gamma y}.$$

Since $\prod_{i\in I}(\theta_j\gamma-\theta_i\delta)=\prod_{i\in I}(\theta_k\gamma-\theta_i\delta)=\kappa_{_N}\mu$, we observe that

$$\prod_{j'\in J}\left(\frac{\theta_j\gamma-\theta_{j'}\delta}{\theta_k\gamma-\theta_{j'}\delta}\right)=\prod_{i\in I}\left(\frac{\theta_j\gamma-\theta_i\delta}{\theta_k\gamma-\theta_i\delta}\right)\prod_{k'\in K}\left(\frac{\theta_k\gamma-\theta_{k'}\delta}{\theta_j\gamma-\theta_{k'}\delta}\right)=\prod_{k'\in K}\left(\frac{\theta_k\gamma-\theta_{k'}\delta}{\theta_j\gamma-\theta_{k'}\delta}\right).$$

Therefore,

$$\sum_{k\in K}\frac{\theta_kB_k}{\theta_j-\theta_k}\prod_{j'\in J}\left(\frac{\theta_j\gamma-\theta_{j'}\delta}{\theta_k\gamma-\theta_{j'}\delta}\right)=\frac{1}{\prod_{k'\in K}(\theta_j\gamma-\theta_{k'}\delta)}\sum_{k\in K}\frac{\theta_kB_k}{\theta_j-\theta_k}\prod_{k'\in K}(\theta_k\gamma-\theta_{k'}\delta).$$

As previously, we invoke the formula

$$\sum_{k\in K}\frac{\theta_kB_kP(\theta_k)}{\theta_k-X}=(-1)^{\#K}\frac{P(X)}{\prod_{k\in K}(\theta_k-X)}-c$$

which is valid for any polynomial P of degree $\#K$ and coefficient of highest degree c. We obtain for $P(X)=\prod_{k'\in K}(\gamma X-\theta_{k'}\delta)$ and $X=\theta_j$, by using (2.6),

$$\sum_{k\in K}\frac{\theta_kB_k}{\theta_j-\theta_k}\prod_{k'\in K}(\theta_k\gamma-\theta_{k'}\delta)=\gamma^{\#K}-\frac{1}{NA_j}\prod_{k'\in K}(\theta_j\gamma-\theta_{k'}\delta)$$

and then

$$\sum_{k\in K}\frac{\theta_kB_k}{\theta_j-\theta_k}\prod_{j'\in J}\left(\frac{\theta_j\gamma-\theta_{j'}\delta}{\theta_k\gamma-\theta_{j'}\delta}\right)+\frac{1}{NA_j}=\frac{\gamma^{\#K}}{\prod_{k\in K}(\theta_j\gamma-\theta_k\delta)}.$$

Now, expanding the rational fraction $1/(\prod_{k\in K}(\theta_j\gamma-\theta_k\delta))$ into partial fractions yields

$$\frac{1}{\prod_{k\in K}(\theta_j\gamma-\theta_k\delta)}=-\frac{1}{\delta^{\#K-1}}\sum_{k\in K}\frac{\theta_kB_k}{\theta_j\gamma-\theta_k\delta}$$

and finally

$$\varphi_{\lambda,\mu}(0,y)=-\frac{1}{\gamma^{\#J-1}\delta^{\#K-1}}\sum_{j\in J}\theta_jA_j\left(\sum_{k\in K}\frac{\theta_kB_k}{\theta_j\gamma-\theta_k\delta}\right)\mathrm{e}^{-\theta_j\gamma y}.$$

We have checked (3.10). ■

C Proof of (5.7)

Set, for any integer m such that $m\le N-1$ and any complex number β,

$$F_m(s,t;\xi,\beta)=t^{\frac1N-1}\int_0^s I_m(s-\sigma;\xi)\,f(\sigma;\beta\sqrt[N]{t}\,)\,\mathrm{d}\sigma,$$

$$G_m(s,t;\xi,\beta)=\int_0^s\sigma^{\frac1N-1}I_m(s-\sigma;\xi)\,\mathrm{d}\sigma\int_0^t\frac{(t-\tau)^{\frac1N-1}}{\Gamma\left(\frac1N\right)}f(\tau;\beta\sqrt[N]{\sigma}\,)\,\mathrm{d}\tau.$$

Let us compute the Laplace transforms of $F_m(s,t;\xi,\beta)$ and $G_m(s,t;\xi,\beta)$. For this, we need the result below concerning the Laplace transform of the Mittag-Leffler function.

Lemma C.1 *The following identity holds for $\lambda,\mu>0$ and $\beta\in\mathbb{C}$ such that $\frac{\pi}{2N}<\arg(\beta)<2\pi-\frac{\pi}{2N}$:*

$$\int_0^\infty\mathrm{e}^{-\lambda t}t^{\frac1N-1}E_{\frac1N,\frac1N}\left(\beta\sqrt[N]{\mu t}\,\right)\mathrm{d}t=\frac{1}{\sqrt[N]{\lambda}-\beta\sqrt[N]{\mu}}.$$

Proof

Referring to the asymptotics $E_{\frac1N,\frac1N}(z)\sim-1/[\Gamma\left(-\frac1N\right)z^2]$ when z tends to infinity such that $|\arg(-z)|<(1-\frac{1}{2N})\pi$ (see Formula (21), p. 210 of [7]), the condition $\frac{\pi}{2N}<\arg(\beta)<2\pi-\frac{\pi}{2N}$ makes sure the absolute convergence of the integral lying in the statement of Lemma (C.1).

First, suppose that $\lambda > \mu\,|\beta|^N$. In this case we easily have

$$\int_0^\infty \mathrm{e}^{-\lambda t} t^{\frac{1}{N}-1} E_{\frac{1}{N},\frac{1}{N}}\big(\beta\sqrt[N]{\mu t}\,\big)\,\mathrm{d}t = \sum_{n=0}^{\infty} \frac{\big(\beta\sqrt[N]{\mu}\,\big)^n}{\Gamma\big(\frac{n+1}{N}\big)} \int_0^\infty \mathrm{e}^{-\lambda t} t^{\frac{n+1}{N}-1}\,\mathrm{d}t$$

$$= \sum_{n=0}^{\infty} \frac{\big(\beta\sqrt[N]{\mu}\,\big)^n}{\lambda^{\frac{n+1}{N}}} = \frac{1}{\sqrt[N]{\lambda}-\beta\sqrt[N]{\mu}}.$$

Suppose now that $\lambda < \mu\,|\beta|^N$. Recalling that

$$E_{\frac{1}{N},\frac{1}{N}}\big(\beta\sqrt[N]{\mu t}\,\big) = \frac{1}{\sqrt[N]{\mu}} \int_0^\infty \mathrm{e}^{-\mu s} f\big(s;\beta\sqrt[N]{t}\,\big)\,\mathrm{d}s$$

with

$$f(s;\beta) = \frac{\sin(\frac{\pi}{N})}{\pi N}\, \frac{s^{\frac{1}{N}-1}(\beta^2 - s^{\frac{2}{N}})}{(s^{\frac{2}{N}} - 2\beta\cos(\frac{\pi}{N})s^{\frac{1}{N}} + \beta^2)^2},$$

we obtain

$$\int_0^\infty \mathrm{e}^{-\lambda t} t^{\frac{1}{N}-1} E_{\frac{1}{N},\frac{1}{N}}\big(\beta\sqrt[N]{\mu t}\,\big)\,\mathrm{d}t = \frac{1}{\sqrt[N]{\mu}} \int_0^\infty \mathrm{e}^{-\lambda t} t^{\frac{1}{N}-1} \left(\int_0^\infty \mathrm{e}^{-\mu s} f\big(s;\beta\sqrt[N]{t}\,\big)\,\mathrm{d}s\right)\mathrm{d}t.$$

It may be easily seen that the double integral $\int_0^\infty\!\int_0^\infty \mathrm{e}^{-\lambda t-\mu s} t^{\frac{1}{N}-1} f\big(s;\beta\sqrt[N]{t}\,\big)\,\mathrm{d}s\,\mathrm{d}t$ is not absolutely convergent (because of its behavior near $(0,0)$). So, we can not interchange the integrals and we must excise one integral near zero as follows:

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-\lambda t} t^{\frac{1}{N}-1} \left(\int_0^\infty \mathrm{e}^{-\mu s} f\big(s;\beta\sqrt[N]{t}\,\big)\,\mathrm{d}s\right)\mathrm{d}t = \lim_{\varepsilon\to 0^+} \Bigg[&\int_0^\infty \mathrm{e}^{-\lambda t} t^{\frac{1}{N}-1} \left(\int_0^{\varepsilon^N} \mathrm{e}^{-\mu s} f\big(s;\beta\sqrt[N]{t}\,\big)\,\mathrm{d}s\right)\mathrm{d}t \\ &+ \int_0^\infty \mathrm{e}^{-\lambda t} t^{\frac{1}{N}-1} \left(\int_{\varepsilon^N}^\infty \mathrm{e}^{-\mu s} f\big(s;\beta\sqrt[N]{t}\,\big)\,\mathrm{d}s\right)\mathrm{d}t\Bigg]. \qquad \text{(C.1)}\end{aligned}$$

We begin by evaluating the first term lying on the right-hand side of (C.1):

$$\begin{aligned}&\lim_{\varepsilon\to 0^+} \int_0^\infty \mathrm{e}^{-\lambda t} t^{\frac{1}{N}-1} \left(\int_0^{\varepsilon^N} \mathrm{e}^{-\mu s} f\big(s;\beta\sqrt[N]{t}\,\big)\,\mathrm{d}s\right)\mathrm{d}t \\ &\quad= N^2\varepsilon^{N+1} \lim_{\varepsilon\to 0^+} \int_0^\infty \mathrm{e}^{-\lambda\varepsilon^N t^N} \left(\int_0^1 \mathrm{e}^{-\mu\varepsilon^N s^N} s^{N-1} f\big(\varepsilon^N s^N;\beta\varepsilon t\big)\,\mathrm{d}s\right)\mathrm{d}t \\ &\quad= \frac{N}{\pi}\sin\Big(\frac{\pi}{N}\Big) \lim_{\varepsilon\to 0^+} \int_0^\infty \mathrm{e}^{-\lambda\varepsilon^N t^N} \left(\int_0^1 \mathrm{e}^{-\mu\varepsilon^N s^N} \frac{\beta^2 t^2 - s^2}{(s^2 - 2\beta\cos(\frac{\pi}{N})st + \beta^2 t^2)^2}\,\mathrm{d}s\right)\mathrm{d}t \\ &\quad= \frac{N}{\pi}\sin\Big(\frac{\pi}{N}\Big) \int_0^\infty \left(\int_0^1 \frac{\beta^2 t^2 - s^2}{(s^2 - 2\beta\cos(\frac{\pi}{N})st + \beta^2 t^2)^2}\,\mathrm{d}s\right)\mathrm{d}t.\end{aligned}$$

The integral with respect to the variable s is elementary:

$$\begin{aligned}\int_0^1 \frac{\beta^2 t^2 - s^2}{(s^2 - 2\beta\cos(\frac{\pi}{N})st + \beta^2 t^2)^2}\,\mathrm{d}s &= \left[\frac{s}{s^2 - 2\beta\cos(\frac{\pi}{N})st + \beta^2 t^2}\right]_{s=0}^{s=1} \\ &= \frac{1}{1 - 2\beta\cos(\frac{\pi}{N})t + \beta^2 t^2} = \frac{1}{[\beta t - \cos(\frac{\pi}{N})]^2 + \sin^2(\frac{\pi}{N})}\end{aligned}$$

and next

$$\int_0^\infty \left(\int_0^1 \frac{\beta^2 t^2 - s^2}{(s^2 - 2\beta\cos(\frac{\pi}{N})st + \beta^2 t^2)^2}\,\mathrm{d}s\right)\mathrm{d}t = \frac{1}{\beta\sin(\frac{\pi}{N})} \left[\operatorname{arccot}\left(\frac{\cos(\frac{\pi}{N}) - \beta t}{\sin(\frac{\pi}{N})}\right)\right]_{t=0}^{t=\infty} = -\frac{\pi}{N\beta\sin(\frac{\pi}{N})}.$$

Consequently,

$$\lim_{\varepsilon\to 0^+} \int_0^\infty \mathrm{e}^{-\lambda t} t^{\frac{1}{N}-1} \left(\int_0^{\varepsilon^N} \mathrm{e}^{-\mu s} f\big(s;\beta\sqrt[N]{t}\,\big)\,\mathrm{d}s\right)\mathrm{d}t = -\frac{1}{\beta}. \qquad \text{(C.2)}$$

Concerning the second term lying on the right-hand side of (C.1), since

$$f\big(s;\beta\sqrt[N]{t}\,\big) = -\frac{1}{\beta^2}\Big(\frac{s}{t}\Big)^{\frac{1}{N}-1} f\Big(t;\frac{1}{\beta}\sqrt[N]{s}\,\Big),$$

we have (in this case, interchanging the two integrals is valid)

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\int_0^\infty \mathrm{e}^{-\lambda t}t^{\frac{1}{N}-1}\left(\int_{\varepsilon^N}^\infty \mathrm{e}^{-\mu s}f\big(s;\beta\sqrt[N]{t}\,\big)\,\mathrm{d}s\right)\mathrm{d}t &= -\frac{1}{\beta^2}\lim_{\varepsilon\to 0^+}\int_{\varepsilon^N}^\infty\!\!\int_0^\infty \mathrm{e}^{-\lambda t-\mu s}s^{\frac{1}{N}-1}f\left(t;\frac{1}{\beta}\sqrt[N]{s}\right)\mathrm{d}s\,\mathrm{d}t\\
&= -\frac{\sqrt[N]{\lambda}}{\beta^2}\lim_{\varepsilon\to 0^+}\int_{\varepsilon^N}^\infty \mathrm{e}^{-\mu s}s^{\frac{1}{N}-1}E_{\frac{1}{N},\frac{1}{N}}\left(\frac{1}{\beta}\sqrt[N]{\lambda s}\right)\mathrm{d}s\\
&= -\frac{\sqrt[N]{\lambda}}{\beta^2}\int_0^\infty \mathrm{e}^{-\mu s}s^{\frac{1}{N}-1}E_{\frac{1}{N},\frac{1}{N}}\left(\frac{1}{\beta}\sqrt[N]{\lambda s}\right)\mathrm{d}s\\
&= -\frac{\sqrt[N]{\lambda}}{\beta^2}\,\frac{1}{\sqrt[N]{\mu}-\frac{1}{\beta}\sqrt[N]{\lambda}} = \frac{1}{\beta}\,\frac{\sqrt[N]{\lambda}}{\sqrt[N]{\lambda}-\beta\sqrt[N]{\mu}}. \qquad \text{(C.3)}\end{aligned}$$

In the last step, we have used the result corresponding to the first case. Finally, adding (C.2) to (C.3) yields the statement in the case where $\lambda<\mu|\beta|^N$. ■

We can compute, with the help of Lemma C.1, the Laplace transforms of $F_m(s,t;\xi,\beta)$ and $G_m(s,t;\xi,\beta)$. On the one hand,

$$\begin{aligned}\int_0^\infty\!\!\int_0^\infty \mathrm{e}^{-\lambda t-\mu s}F_m(s,t;\xi,\beta)\,\mathrm{d}s\,\mathrm{d}t &= \int_0^\infty \mathrm{e}^{-\lambda t}t^{\frac{1}{N}-1}\left[\int_0^\infty \mathrm{e}^{-\mu s}I_m(s;\xi)\,\mathrm{d}s\int_0^\infty \mathrm{e}^{-\mu s}f\big(s;\beta\sqrt[N]{t}\,\big)\,\mathrm{d}s\right]\mathrm{d}t\\
&= \frac{\mathrm{e}^{-\xi\sqrt[N]{\mu}}}{\mu^{\frac{m-1}{N}}}\int_0^\infty \mathrm{e}^{-\lambda t}t^{\frac{1}{N}-1}E_{\frac{1}{N},\frac{1}{N}}\big(\beta\sqrt[N]{\mu t}\,\big)\,\mathrm{d}t\\
&= \frac{\mathrm{e}^{-\xi\sqrt[N]{\mu}}}{\mu^{\frac{m-1}{N}}\big(\sqrt[N]{\lambda}-\beta\sqrt[N]{\mu}\,\big)}.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\int_0^\infty\!\!\int_0^\infty \mathrm{e}^{-\lambda t-\mu s}G_m(s,t;\xi,\beta)\,\mathrm{d}s\,\mathrm{d}t\\
&\quad= \int_0^\infty \mathrm{e}^{-\mu s}I_m(s;\xi)\,\mathrm{d}s\int_0^\infty \mathrm{e}^{-\mu s}s^{\frac{1}{N}-1}\left[\int_0^\infty \mathrm{e}^{-\lambda t}\frac{t^{\frac{1}{N}-1}}{\Gamma\big(\frac{1}{N}\big)}\,\mathrm{d}s\int_0^\infty \mathrm{e}^{-\lambda t}f\big(t;\beta\sqrt[N]{s}\,\big)\,\mathrm{d}t\right]\mathrm{d}s\\
&\quad= \frac{\mathrm{e}^{-\xi\sqrt[N]{\mu}}}{\mu^{\frac{m}{N}}}\int_0^\infty \mathrm{e}^{-\mu s}s^{\frac{1}{N}-1}E_{\frac{1}{N},\frac{1}{N}}\big(\beta\sqrt[N]{\lambda s}\,\big)\,\mathrm{d}s\\
&\quad= \frac{\mathrm{e}^{-\xi\sqrt[N]{\mu}}}{\mu^{\frac{m}{N}}\big(\sqrt[N]{\mu}-\beta\sqrt[N]{\lambda}\,\big)}.\end{aligned}$$

Introducing back the settings $\gamma=\sqrt[N]{\lambda+\mu}$ and $\delta=\sqrt[N]{\lambda}$, the Laplace transforms of each member of (5.7) can be evaluated as follows. We choose $\beta=\theta_{j'}/\theta_k$ or $\theta_k/\theta_{j'}$ with $j'\in J$, $k\in K$; in both cases, we have $\frac{\pi}{2N}<\arg(\beta)<2\pi-\frac{\pi}{2N}$. Thus we can use the above results and we obtain

$$\begin{aligned}&\int_0^\infty\!\!\int_s^\infty \mathrm{e}^{-\lambda t-\mu s}\sum_{j,j'\in J,\,k\in K}\frac{\theta_jA_jA_{j'}B_kC_{jj'k}}{\theta_k^{\#J-1}(\theta_j-\theta_k)}\\
&\qquad\times\left[\int_0^s \sigma^{\frac{1}{N}-1}I_{N-2}(s-\sigma;\theta_jy-\theta_kx)\,\mathrm{d}\sigma\int_0^{t-s}\frac{(t-s-\tau)^{\frac{1}{N}-1}}{\Gamma\big(\frac{1}{N}\big)}f\left(\tau;\frac{\theta_{j'}}{\theta_k}\sqrt[N]{\sigma}\right)\mathrm{d}\tau\right]\mathrm{d}s\,\mathrm{d}t\\
&\quad= \sum_{j,j'\in J,\,k\in K}\frac{\theta_jA_jA_{j'}B_kC_{jj'k}}{\theta_k^{\#J-1}(\theta_j-\theta_k)}\int_0^\infty\!\!\int_s^\infty \mathrm{e}^{-\lambda t-\mu s}\,G_{N-2}\left(s,t-s;\theta_jy-\theta_kx,\frac{\theta_{j'}}{\theta_k}\right)\mathrm{d}s\,\mathrm{d}t\\
&\quad= \sum_{j,j'\in J,\,k\in K}\frac{\theta_jA_jA_{j'}B_kC_{jj'k}}{\theta_k^{\#J-1}(\theta_j-\theta_k)}\int_0^\infty\!\!\int_0^\infty \mathrm{e}^{-\lambda t-(\lambda+\mu)s}\,G_{N-2}\left(s,t;\theta_jy-\theta_kx,\frac{\theta_{j'}}{\theta_k}\right)\mathrm{d}s\,\mathrm{d}t\\
&\quad= \frac{1}{\gamma^{N-2}}\sum_{j,j'\in J,\,k\in K}\frac{\theta_jA_jA_{j'}B_kC_{jj'k}}{\theta_k^{\#J-2}(\theta_j-\theta_k)}\,\frac{\mathrm{e}^{\gamma(\theta_kx-\theta_jy)}}{\theta_k\gamma-\theta_{j'}\delta}. \qquad \text{(C.4)}\end{aligned}$$

Similarly,

$$\int_0^\infty\!\!\int_s^\infty \mathrm{e}^{-\lambda t-\mu s}\sum_{j,j'\in J,\,k\in K}\frac{\theta_jA_jA_{j'}B_kC_{jj'k}}{\theta_{j'}\theta_k^{\#J-2}(\theta_j-\theta_k)}$$

$$\times\left[(t-s)^{\frac{1}{N}-1}\int_0^s I_{N-1}(s-\sigma;\theta_j y-\theta_k x)\, f\Big(\sigma;\frac{\theta_k}{\theta_{j'}}\sqrt[N]{t-s}\Big)\,\mathrm{d}\sigma\right]\mathrm{d}s\,\mathrm{d}t$$

$$=\sum_{j,j'\in J,\,k\in K}\frac{\theta_j A_j A_{j'} B_k C_{jj'k}}{\theta_{j'}\theta_k^{\#J-2}(\theta_j-\theta_k)}\int_0^{\infty}\!\!\int_s^{\infty}\mathrm{e}^{-\lambda t-\mu s}\,F_{N-1}\Big(s,t-s;\theta_j y-\theta_k x,\frac{\theta_k}{\theta_{j'}}\Big)\,\mathrm{d}s\,\mathrm{d}t$$

$$=\sum_{j,j'\in J,\,k\in K}\frac{\theta_j A_j A_{j'} B_k C_{jj'k}}{\theta_{j'}\theta_k^{\#J-2}(\theta_j-\theta_k)}\int_0^{\infty}\!\!\int_0^{\infty}\mathrm{e}^{-\lambda t-(\lambda+\mu)s}\,F_{N-1}\Big(s,t;\theta_j y-\theta_k x,\frac{\theta_k}{\theta_{j'}}\Big)\,\mathrm{d}s\,\mathrm{d}t$$

$$=-\frac{1}{\gamma^{N-2}}\sum_{j,j'\in J,\,k\in K}\frac{\theta_j A_j A_{j'} B_k C_{jj'k}}{\theta_k^{\#J-2}(\theta_j-\theta_k)}\,\frac{\mathrm{e}^{\gamma(\theta_k x-\theta_j y)}}{\theta_k\gamma-\theta_{j'}\delta}. \tag{C.5}$$

We see that the quantities (C.4) and (C.5) are opposite which completes the proof of (5.7). ■

Addendum. In [5], some constants in Formulas (2.8) and (2.9) can be simplified: since $\kappa_N=(-1)^{\#J-1}$, $\prod_{j\in J}\theta_j=1$ and $\prod_{k\in K}\theta_k=(-1)^{\#K}$, we have $\alpha_{-\#K}=-\beta_{\#K}=1$ and $\alpha_{1-\#K}=\beta_{\#K+1}=\sum_{j\in J}\theta_j$.

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